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Joining of Decorated Trees

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Summary. This is the continuation of the sequence of articles on trees (see [3,4,5]). The main goal is to introduce joining operations on decorated trees corresponding with operations introduced in [5]. We will also introduce the operation of substitution. In the last section we dealt with trees decorated by Cartesian product, i.e. we showed some lemmas on joining operations applied to such trees.

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The notation and terminology used here are introduced in the following papers: [15], [2], [9], [16], [11], [14], [13], [12], [10], [7], [6], [8], [3], [4], [1], and [5].

1. Joining of Decorated Tree

Let T be a decorated tree. A node of T is an element of $\operatorname{dom} T$.

We adopt the following convention: x, y, z are arbitrary, i, j, n denote natural numbers, and p, q denote finite sequences.

Let T_1, T_2 be decorated trees. Let us observe that $T_1 = T_2$ if and only if:

(Def.1) $\operatorname{dom} T_1 = \operatorname{dom} T_2$ and for every node p of T_1 holds $T_1(p) = T_2(p)$.

One can prove the following two propositions:

(1) For all natural numbers i, j such that the elementary tree of $i \subseteq$ the elementary tree of j holds $i \leq j$.

(2) For all natural numbers i, j such that the elementary tree of $i =$ the elementary tree of j holds $i = j$.

Let us consider x. The root tree of x is a decorated tree and is defined as follows:

(Def.2) The root tree of $x =$ (the elementary tree of $0) \longmapsto x$.

Let D be a non empty set and let d be an element of D. Then the root tree of d is an element of $\operatorname{FinTrees}(D)$.

We now state four propositions:

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(3) dom (the root tree of x) = the elementary tree of 0 and (the root tree of x)(ε) = x.

(4) If the root tree of x = the root tree of y, then $x = y$.

(5) For every decorated tree T such that dom T = the elementary tree of 0 holds T = the root tree of $T(\varepsilon)$.

(6) The root tree of $x = \{\langle\varepsilon, x\rangle\}$.

Let us consider x and let p be a finite sequence. The flat tree of x and p is a decorated tree and is defined by the conditions (Def.3).

(Def.3) (i) dom (the flat tree of x and p) = the elementary tree of len p,

(ii) (the flat tree of x and p)(ε) = x, and

(iii) for every n such that $n <$ len p holds (the flat tree of x and p)($\langle n\rangle$) = $p(n+1)$.

The following propositions are true:

(7) If the flat tree of x and p = the flat tree of y and q, then $x = y$ and $p = q$.

(8) If $j < i$, then (the elementary tree of i) $\restriction \langle j\rangle$ = the elementary tree of 0.

(9) If $i <$ len p, then (the flat tree of x and p) $\restriction \langle i\rangle$ = the root tree of $p(i+1)$.

Let us consider x, p. Let us assume that p is decorated tree yielding. The functor x-tree(p) yields a decorated tree and is defined by the conditions (Def.4).

(Def.4) (i) There exists a decorated tree yielding finite sequence q such that $p = q$ and dom(x-tree(p)) = $\overbrace{\operatorname{dom}_{\kappa} q(\kappa)}$,

(ii) (x-tree(p))(ε) = x, and

(iii) for every n such that $n <$ len p holds (x-tree(p)) $\restriction \langle n\rangle = p(n+1)$.

Let us consider x and let T be a decorated tree. The functor x-tree(T) yielding a decorated tree is defined by:

(Def.5) x-tree(T) = x-tree($\langle T\rangle$).

Let us consider x and let T_1, T_2 be decorated trees. The functor x-tree(T_1, T_2) yields a decorated tree and is defined as follows:

(Def.6) x-tree(T_1, T_2) = x-tree($\langle T_1, T_2\rangle$).

We now state a number of propositions:

(10) For every decorated tree yielding finite sequence p holds dom(x-tree(p)) = $\overbrace{\operatorname{dom}_{\kappa} p(\kappa)}$.

(11) Let p be a decorated tree yielding finite sequence. Then $y \in$ dom(x-tree(p)) if and only if one of the following conditions is satisfied:

(i) $y = \varepsilon$, or

(ii) there exists a natural number i and there exists a decorated tree T and there exists a node q of T such that $i <$ len p and $T = p(i+1)$ and $y = \langle i\rangle \frown q$.

(12) Let p be a decorated tree yielding finite sequence, and let i be a natural number, and let T be a decorated tree, and let q be a node of T. If

$i < \operatorname{len} p$ and $T = p(i+1)$, then $(x\text{-tree}(p))(\langle i \rangle \frown q) = T(q)$.

(13) For every decorated tree T holds $\operatorname{dom}(x\text{-tree}(T)) = \widehat{\operatorname{dom} T}$.

(14) For all decorated trees T_1, T_2 holds $\operatorname{dom}(x\text{-tree}(T_1, T_2)) = \widehat{\operatorname{dom} T_1, \operatorname{dom} T_2}$.

(15) For all decorated tree yielding finite sequence p, q such that $x\text{-tree}(p) = y\text{-tree}(q)$ holds $x = y$ and $p = q$.

(16) If the root tree of x = the flat tree of y and p, then $x = y$ and $p = \varepsilon$.

(17) If the root tree of $x = y\text{-tree}(p)$ and p is decorated tree yielding, then $x = y$ and $p = \varepsilon$.

(18) Suppose the flat tree of x and $p = y\text{-tree}(q)$ and q is decorated tree yielding. Then $x = y$ and $\operatorname{len} p = \operatorname{len} q$ and for every i such that $i \in \operatorname{dom} p$ holds $q(i)$ = the root tree of $p(i)$.

(19) Let p be a decorated tree yielding finite sequence, and let n be a natural number, and let q be a finite sequence. If $\langle n \rangle \frown q \in \operatorname{dom}(x\text{-tree}(p))$, then $(x\text{-tree}(p))(\langle n \rangle \frown q) = p(n+1)(q)$.

(20) The flat tree of x and ε = the root tree of x and $x\text{-tree}(\varepsilon)$ = the root tree of x.

(21) The flat tree of x and $\langle y \rangle$ = ((the elementary tree of 1) $\longmapsto x$)($\langle 0 \rangle$/(the root tree of y)).

(22) For every decorated tree T holds $x\text{-tree}(\langle T \rangle)$ = ((the elementary tree of 1) $\longmapsto x$)($\langle 0 \rangle / T$).

Let D be a non empty set, let d be an element of D, and let p be a finite sequence of elements of D. Then the flat tree of d and p is a tree decorated by D.

Let D be a non empty set, let F be a non empty set of trees decorated by D, let d be an element of D, and let p be a finite sequence of elements of F. Then $d\text{-tree}(p)$ is a tree decorated by D.

Let D be a non empty set, let d be an element of D, and let T be a tree decorated by D. Then $d\text{-tree}(T)$ is a tree decorated by D.

Let D be a non empty set, let d be an element of D, and let T_1, T_2 be trees decorated by D. Then $d\text{-tree}(T_1, T_2)$ is a tree decorated by D.

Let D be a non empty set and let p be a finite sequence of elements of FinTrees(D). Then $\operatorname{dom}_\kappa p(\kappa)$ is a finite sequence of elements of FinTrees.

Let D be a non empty set, let d be an element of D, and let p be a finite sequence of elements of FinTrees(D). Then $d\text{-tree}(p)$ is an element of FinTrees(D).

Let D be a non empty set and let x be a subset of D. We see that the finite sequence of elements of x is a finite sequence of elements of D.

Let D be a non empty constituted of decorated trees set and let X be a subset of D. Note that every finite sequence of elements of X is decorated tree yielding.

2. Expanding of Decorated Tree by Substitution

The scheme *ExpandTree* concerns a tree $\mathcal{A}$, a tree $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a tree T such that for every p holds $p \in T$ if and only if one of the following conditions is satisfied:

(i) $p \in \mathcal{A}$, or

(ii) there exists an element q of $\mathcal{A}$ and there exists an element r of $\mathcal{B}$ such that $\mathcal{P}[q]$ and $p = q \frown r$

for all values of the parameters.

Let T, T' be decorated trees and let x be arbitrary. The functor $T_{x \leftarrow T'}$ yielding a decorated tree is defined by the conditions (Def.7).

(Def.7) (i) For every p holds $p \in \mathrm{dom}(T_{x \leftarrow T'})$ iff $p \in \mathrm{dom}\, T$ or there exists a node q of T and there exists a node r of T' such that $q \in \mathrm{Leaves}\,\mathrm{dom}\, T$ and $T(q) = x$ and $p = q \frown r$,

(ii) for every node p of T such that $p \notin \mathrm{Leaves}\,\mathrm{dom}\, T$ or $T(p) \neq x$ holds $T_{x \leftarrow T'}(p) = T(p)$, and

(iii) for every node p of T and for every node q of T' such that $p \in \mathrm{Leaves}\,\mathrm{dom}\, T$ and $T(p) = x$ holds $T_{x \leftarrow T'}(p \frown q) = T'(q)$.

Let D be a non empty set, let T, T' be trees decorated by D, and let x be arbitrary. Then $T_{x \leftarrow T'}$ is a tree decorated by D.

We follow a convention: T, T', T_1, T_2 are decorated trees and x, y, z are arbitrary.

One can prove the following proposition

(23) If $x \notin \mathrm{rng}\, T$ or $x \notin \mathrm{Leaves}\, T$, then $T_{x \leftarrow T'} = T$.

3. Double Decorated Trees

For simplicity we adopt the following rules: D_1, D_2 are non empty set, T is a tree decorated by D_1 and D_2, F is a non empty set of trees decorated by D_1 and D_2, and F_1 is a non empty set of trees decorated by D_1.

The following propositions are true:

(24) For all D_1, D_2, T holds $\mathrm{dom}(T_1) = \mathrm{dom}\, T$ and $\mathrm{dom}(T_2) = \mathrm{dom}\, T$.

(25) (the root tree of $\langle d_1, d_2 \rangle)_1$ = the root tree of d_1 and (the root tree of $\langle d_1, d_2 \rangle)_2$ = the root tree of d_2.

(26) $\langle$the root tree of x, the root tree of $y\rangle$ = the root tree of $\langle x, y \rangle$.

(27) Given D_1, D_2, d_1, d_2, F, F_1, and let p be a finite sequence of elements of F, and let p_1 be a finite sequence of elements of F_1. Suppose $\mathrm{dom}\, p_1 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ and for every T such that $T = p(i)$ holds $p_1(i) = T_1$. Then $(\langle d_1, d_2 \rangle\text{-tree}(p))_1 = d_1\text{-tree}(p_1)$.

(28) Given D_1, D_2, d_1, d_2, F, F_2, and let p be a finite sequence of elements of F, and let p_2 be a finite sequence of elements of F_2. Suppose $\mathrm{dom}\, p_2 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ and for every T such that $T = p(i)$ holds $p_2(i) = T_{\mathbf{2}}$. Then $(\langle d_1, d_2\rangle\text{-tree}(p))_{\mathbf{2}} = d_2\text{-tree}(p_2)$.

(29) Given D_1, D_2, d_1, d_2, F and let p be a finite sequence of elements of F. Then there exists a finite sequence p_1 of elements of $\mathrm{Trees}(D_1)$ such that $\mathrm{dom}\, p_1 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ there exists an element T of F such that $T = p(i)$ and $p_1(i) = T_{\mathbf{1}}$ and $(\langle d_1, d_2\rangle\text{-tree}(p))_{\mathbf{1}} = d_1\text{-tree}(p_1)$.

(30) Given D_1, D_2, d_1, d_2, F and let p be a finite sequence of elements of F. Then there exists a finite sequence p_2 of elements of $\mathrm{Trees}(D_2)$ such that $\mathrm{dom}\, p_2 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ there exists an element T of F such that $T = p(i)$ and $p_2(i) = T_{\mathbf{2}}$ and $(\langle d_1, d_2\rangle\text{-tree}(p))_{\mathbf{2}} = d_2\text{-tree}(p_2)$.

(31) Given D_1, D_2, d_1, d_2 and let p be a finite sequence of elements of $\mathrm{FinTrees}([\!:D_1, D_2:\!])$. Then there exists a finite sequence p_1 of elements of $\mathrm{FinTrees}(D_1)$ such that $\mathrm{dom}\, p_1 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ there exists an element T of $\mathrm{FinTrees}([\!:D_1, D_2:\!])$ such that $T = p(i)$ and $p_1(i) = T_{\mathbf{1}}$ and $(\langle d_1, d_2\rangle\text{-tree}(p))_{\mathbf{1}} = d_1\text{-tree}(p_1)$.

(32) Given D_1, D_2, d_1, d_2 and let p be a finite sequence of elements of $\mathrm{FinTrees}([\!:D_1, D_2:\!])$. Then there exists a finite sequence p_2 of elements of $\mathrm{FinTrees}(D_2)$ such that $\mathrm{dom}\, p_2 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ there exists an element T of $\mathrm{FinTrees}([\!:D_1, D_2:\!])$ such that $T = p(i)$ and $p_2(i) = T_{\mathbf{2}}$ and $(\langle d_1, d_2\rangle\text{-tree}(p))_{\mathbf{2}} = d_2\text{-tree}(p_2)$.

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