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Subalgebras of the Universal Algebra. Lattices of Subalgebras

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Summary. Introduces a definition of a subalgebra of a universal algebra. A notion of similar algebras and basic operations on subalgebras such as a subalgebra generated by a set, the intersection and the sum of two subalgebras were introduced. Some basic facts concerning the above notions have been proved. The article also contains the definition of a lattice of subalgebras of a universal algebra.

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The papers [7], [8], [4], [1], [5], [3], [9], [2], and [6] provide the terminology and notation for this paper.

One can prove the following propositions:

(1) For every natural number n and for every non empty set D and for every non empty subset D_1 of D holds $D^n \cap {D_1}^n = {D_1}^n$.

(2) For every non empty set D and for every homogeneous quasi total non empty partial function h from D^* to D holds $\operatorname{dom} h = D^{\operatorname{arity} h}$.

We follow a convention: U_0, U_1, U_2, U_3 denote universal algebras, n, i denote natural numbers, and a denotes an element of the carrier of U_0.

Let D be a non empty set. A non empty set is called a set of universal functions on D if:

(Def.1) Every element of it is a homogeneous quasi total non empty partial function from D^* to D.

Let D be a non empty set and let P be a set of universal functions on D. We see that the element of P is a homogeneous quasi total non empty partial function from D^* to D.

Let us consider U_1. A set of universal functions on U_1 is a set of universal functions on the carrier of U_1.

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Let U_1 be a universal algebra structure. A partial function on U_1 is a partial function from (the carrier of U_1)* to the carrier of U_1.

Let us consider U_1, U_2. We say that U_1 and U_2 are similar if and only if:

(Def.2) signature U_1 = signature U_2.

Let us observe that the predicate introduced above is reflexive symmetric.

The following propositions are true:

(3) If U_1 and U_2 are similar, then len Opers U_1 = len Opers U_2.

(4) If U_1 and U_2 are similar and U_2 and U_3 are similar, then U_1 and U_3 are similar.

(5) rng Opers U_0 is a non empty subset of (the carrier of U_0)*$\dot{\rightarrow}$the carrier of U_0.

Let us consider U_0. The functor Operations(U_0) yielding a set of universal functions on U_0 is defined as follows:

(Def.3) Operations(U_0) = rng Opers U_0.

Let us consider U_1. A operation of U_1 is an element of Operations(U_1).

Let us consider U_0. A subset of U_0 is a subset of the carrier of U_0.

In the sequel x_1, y_1 will denote finite sequences of elements of A.

One can prove the following proposition

(6) If $n \in$ dom Opers U_0, then (Opers U_0)(n) is a operation of U_0.

Let U_0 be a universal algebra, let A be a subset of U_0, and let o be a operation of U_0. We say that A is closed on o if and only if:

(Def.4) For every finite sequence s of elements of A such that len s = arity o holds $o(s) \in A$.

Let U_0 be a universal algebra and let A be a subset of U_0. We say that A is operations closed if and only if:

(Def.5) For every operation o of U_0 holds A is closed on o.

Let us consider U_0, A, o. Let us assume that A is closed on o. The functor o_A yielding a homogeneous quasi total non empty partial function from A^* to A is defined as follows:

(Def.6) $o_A = o \restriction A^{\mathrm{arity}\, o}$.

Let us consider U_0, A. The functor Opers(U_0, A) yields a finite sequence of elements of $A^*\dot{\rightarrow}A$ and is defined as follows:

(Def.7) dom Opers(U_0, A) = dom Opers U_0 and for all n, o such that $n \in$ dom Opers(U_0, A) and o = (Opers U_0)(n) holds (Opers(U_0, A))(n) = o_A.

The following two propositions are true:

(7) For every non empty subset B of U_0 such that B = the carrier of U_0 holds B is operations closed and for every o holds $o_B = o$.

(8) Let U_1 be a universal algebra, and let A be a non empty subset of U_1, and let o be a operation of U_1. If A is closed on o, then arity(o_A) = arity o.

Let us consider U_0. A universal algebra is said to be a subalgebra of U_0 if it satisfies the conditions (Def.8).

(Def.8) (i) The carrier of it is a subset of U_0, and

(ii) for every non empty subset B of U_0 such that $B =$ the carrier of it holds Opers it $=$ Opers(U_0, B) and B is operations closed.

Let U_0 be a universal algebra. One can verify that there exists a subalgebra of U_0 which is strict.

One can prove the following propositions:

(9) Let U_0, U_1 be universal algebras, and let o_0 be a operation of U_0, and let o_1 be a operation of U_1, and let n be a natural number. Suppose U_0 is a subalgebra of U_1 and $n \in \operatorname{dom} \operatorname{Opers} U_0$ and $o_0 = (\operatorname{Opers} U_0)(n)$ and $o_1 = (\operatorname{Opers} U_1)(n)$. Then $\operatorname{arity} o_0 = \operatorname{arity} o_1$.

(10) If U_0 is a subalgebra of U_1, then $\operatorname{dom} \operatorname{Opers} U_0 = \operatorname{dom} \operatorname{Opers} U_1$.

(11) U_0 is a subalgebra of U_0.

(12) If U_0 is a subalgebra of U_1 and U_1 is a subalgebra of U_2, then U_0 is a subalgebra of U_2.

(13) If U_1 is a strict subalgebra of U_2 and U_2 is a strict subalgebra of U_1, then $U_1 = U_2$.

(14) For all subalgebras U_1, U_2 of U_0 such that the carrier of $U_1 \subseteq$ the carrier of U_2 holds U_1 is a subalgebra of U_2.

(15) For all strict subalgebra U_1, U_2 of U_0 such that the carrier of $U_1 =$ the carrier of U_2 holds $U_1 = U_2$.

(16) If U_1 is a subalgebra of U_2, then U_1 and U_2 are similar.

(17) For every non empty subset A of U_0 holds $\langle A, \operatorname{Opers}(U_0, A)\rangle$ is a strict universal algebra.

Let U_0 be a universal algebra and let A be a non empty subset of U_0. Let us assume that A is operations closed. The functor $\langle A, \operatorname{Ops}\rangle$ yielding a strict subalgebra of U_0 is defined as follows:

(Def.9) $\langle A, \operatorname{Ops}\rangle = \langle A, \operatorname{Opers}(U_0, A)\rangle$.

Let us consider U_0 and let U_1, U_2 be subalgebras of U_0. Let us assume that (the carrier of U_1) $\cap$ (the carrier of U_2) $\neq \emptyset$. The functor $U_1 \cap U_2$ yielding a strict subalgebra of U_0 is defined by the conditions (Def.10).

(Def.10) (i) The carrier of $U_1 \cap U_2 =$ (the carrier of U_1) $\cap$ (the carrier of U_2), and

(ii) for every non empty subset B of U_0 such that $B =$ the carrier of $U_1 \cap U_2$ holds $\operatorname{Opers}(U_1 \cap U_2) = \operatorname{Opers}(U_0, B)$ and B is operations closed.

Let us consider U_0. The functor Constants(U_0) yielding a subset of U_0 is defined by:

(Def.11) Constants(U_0) $=$ $\{a$: a ranges over elements of the carrier of U_0, $\exists_o \operatorname{arity} o = 0 \ \wedge \ a \in \operatorname{rng} o\}$.

A universal algebra has constants if:

(Def.12) There exists a operation o of it such that $\operatorname{arity} o = 0$.

Let us note that there exists a universal algebra which is strict and has constants.

Let U_0 be a universal algebra with constants. Then Constants(U_0) is a non empty subset of U_0.

One can prove the following three propositions:

(18) For every universal algebra U_0 and for every subalgebra U_1 of U_0 holds Constants(U_0) is a subset of U_1.

(19) For every universal algebra U_0 with constants and for every subalgebra U_1 of U_0 holds Constants(U_0) is a non empty subset of U_1.

(20) Let U_0 be a universal algebra with constants and let U_1, U_2 be subalgebras of U_0. Then (the carrier of U_1) $\cap$ (the carrier of U_2) $\neq \emptyset$.

Let U_0 be a universal algebra and let A be a subset of U_0. Let us assume that Constants(U_0) $\neq \emptyset$ or $A \neq \emptyset$. The functor $\mathrm{Gen}^{\mathrm{UA}}(A)$ yields a strict subalgebra of U_0 and is defined by the conditions (Def.13).

(Def.13) (i) $A \subseteq$ the carrier of $\mathrm{Gen}^{\mathrm{UA}}(A)$, and

(ii) for every subalgebra U_1 of U_0 such that $A \subseteq$ the carrier of U_1 holds $\mathrm{Gen}^{\mathrm{UA}}(A)$ is a subalgebra of U_1.

Next we state two propositions:

(21) For every strict universal algebra U_0 holds $\mathrm{Gen}^{\mathrm{UA}}(\Omega_{\text{the carrier of } U_0}) = U_0$.

(22) Let U_0 be a universal algebra, and let U_1 be a strict subalgebra of U_0, and let B be a non empty subset of U_0. If $B =$ the carrier of U_1, then $\mathrm{Gen}^{\mathrm{UA}}(B) = U_1$.

Let U_0 be a universal algebra and let U_1, U_2 be subalgebras of U_0. The functor $U_1 \bigsqcup U_2$ yields a strict subalgebra of U_0 and is defined by:

(Def.14) For every non empty subset A of U_0 such that $A =$ (the carrier of U_1) $\cup$ (the carrier of U_2) holds $U_1 \bigsqcup U_2 = \mathrm{Gen}^{\mathrm{UA}}(A)$.

Next we state four propositions:

(23) Let U_0 be a universal algebra, and let U_1 be a subalgebra of U_0, and let A, B be subsets of U_0. If $A \neq \emptyset$ or Constants(U_0) $\neq \emptyset$ and if $B = A \cup$ the carrier of U_1, then $\mathrm{Gen}^{\mathrm{UA}}(A) \bigsqcup U_1 = \mathrm{Gen}^{\mathrm{UA}}(B)$.

(24) For every universal algebra U_0 and for all subalgebras U_1, U_2 of U_0 holds $U_1 \bigsqcup U_2 = U_2 \bigsqcup U_1$.

(25) For every universal algebra U_0 with constants and for all strict subalgebra U_1, U_2 of U_0 holds $U_1 \cap (U_1 \bigsqcup U_2) = U_1$.

(26) For every universal algebra U_0 with constants and for all strict subalgebra U_1, U_2 of U_0 holds $U_1 \cap U_2 \bigsqcup U_2 = U_2$.

Let U_0 be a universal algebra. The functor Subalgebras(U_0) yields a non empty set and is defined as follows:

(Def.15) For every x holds $x \in$ Subalgebras(U_0) iff x is a strict subalgebra of U_0.

Let U_0 be a universal algebra. The functor $\bigsqcup_{U_0}$ yielding a binary operation on Subalgebras(U_0) is defined by:

(Def.16) For all elements x, y of Subalgebras(U_0) and for all strict subalgebra U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds $\bigsqcup_{(U_0)}(x, y) = U_1 \bigsqcup U_2$.

Let U_0 be a universal algebra. The functor $\sqcap_{U_0}$ yields a binary operation on Subalgebras(U_0) and is defined by:

(Def.17) For all elements x, y of Subalgebras(U_0) and for all strict subalgebra U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds $\sqcap_{(U_0)}(x, y) = U_1 \cap U_2$.

One can prove the following four propositions:

(27) $\sqcup_{(U_0)}$ is commutative.

(28) $\sqcup_{(U_0)}$ is associative.

(29) For every universal algebra U_0 with constants holds $\sqcap_{(U_0)}$ is commutative.

(30) For every universal algebra U_0 with constants holds $\sqcap_{(U_0)}$ is associative.

Let U_0 be a universal algebra with constants. The lattice of subalgebras of U_0 yielding a strict lattice is defined as follows:

(Def.18) The lattice of subalgebras of $U_0 = \langle \text{Subalgebras}(U_0), \sqcup_{(U_0)}, \sqcap_{(U_0)} \rangle$.

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