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On the Group of Inner Automorphisms

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The notation and terminology used in this paper are introduced in the following articles: [6], [2], [3], [1], [5], [11], [4], [9], [10], [7], [8], and [12].

For simplicity we adopt the following rules: G denotes a strict group, H denotes a subgroup of G, a, b, x denote elements of G, and h denotes a homomorphism from G to G.

One can prove the following proposition

(1) For all a, b such that b is an element of H holds $b^a \in H$ iff H is normal.

Let us consider G. One can verify that $\mathrm{Z}(G)$ is normal.

Let us consider G. The functor $\mathrm{Aut}(G)$ yields a non empty set of functions from the carrier of G to the carrier of G and is defined as follows:

(Def.1) Every element of $\mathrm{Aut}(G)$ is a homomorphism from G to G and for every h holds $h \in \mathrm{Aut}(G)$ iff h is one-to-one and an epimorphism.

We now state several propositions:

(2) For every h holds $h \in \mathrm{Aut}(G)$ iff h is one-to-one and an epimorphism.

(3) $\mathrm{Aut}(G) \subseteq (\text{the carrier of } G)^{\text{the carrier of } G}$.

(4) $\mathrm{id}_{(\text{the carrier of } G)}$ is an element of $\mathrm{Aut}(G)$.

(5) For every h holds $h \in \mathrm{Aut}(G)$ iff h is an isomorphism.

(6) For every element f of $\mathrm{Aut}(G)$ holds f^{-1} is a homomorphism from G to G.

(7) For every element f of $\mathrm{Aut}(G)$ holds f^{-1} is an element of $\mathrm{Aut}(G)$.

(8) For all elements f_1, f_2 of $\mathrm{Aut}(G)$ holds $f_1 \cdot f_2$ is an element of $\mathrm{Aut}(G)$.

Let us consider G. The functor $\mathrm{AutComp}(G)$ yielding a binary operation on $\mathrm{Aut}(G)$ is defined as follows:

(Def.2) For all elements x, y of $\mathrm{Aut}(G)$ holds $(\mathrm{AutComp}(G))(x, y) = x \cdot y$.

Let us consider G. The functor $\mathrm{AutGroup}(G)$ yields a strict group and is defined by:

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(Def.3) $\mathrm{AutGroup}(G) = \langle \mathrm{Aut}(G), \mathrm{AutComp}(G) \rangle$.

The following three propositions are true:

(9) For all elements x, y of $\mathrm{AutGroup}(G)$ and for all elements f, g of $\mathrm{Aut}(G)$ such that $x = f$ and $y = g$ holds $x \cdot y = f \cdot g$.

(10) $\mathrm{id}_{(\text{the carrier of } G)} = 1_{\mathrm{AutGroup}(G)}$.

(11) For every element f of $\mathrm{Aut}(G)$ and for every element g of $\mathrm{AutGroup}(G)$ such that $f = g$ holds $f^{-1} = g^{-1}$.

Let us consider G. The functor $\mathrm{InnAut}(G)$ yields a non empty set of functions from the carrier of G to the carrier of G and is defined by the condition (Def.4).

(Def.4) Let f be an element of $(\text{the carrier of } G)^{\text{the carrier of } G}$. Then $f \in \mathrm{InnAut}(G)$ if and only if there exists a such that for every x holds $f(x) = x^a$.

Next we state several propositions:

(12) $\mathrm{InnAut}(G) \subseteq (\text{the carrier of } G)^{\text{the carrier of } G}$.

(13) Every element of $\mathrm{InnAut}(G)$ is an element of $\mathrm{Aut}(G)$.

(14) $\mathrm{InnAut}(G) \subseteq \mathrm{Aut}(G)$.

(15) For all elements f, g of $\mathrm{InnAut}(G)$ holds $(\mathrm{AutComp}(G))(f, g) = f \cdot g$.

(16) $\mathrm{id}_{(\text{the carrier of } G)}$ is an element of $\mathrm{InnAut}(G)$.

(17) For every element f of $\mathrm{InnAut}(G)$ holds f^{-1} is an element of $\mathrm{InnAut}(G)$.

(18) For all elements f, g of $\mathrm{InnAut}(G)$ holds $f \cdot g$ is an element of $\mathrm{InnAut}(G)$.

Let us consider G. The functor $\mathrm{InnAutGroup}(G)$ yields a normal strict subgroup of $\mathrm{AutGroup}(G)$ and is defined by:

(Def.5) The carrier of $\mathrm{InnAutGroup}(G) = \mathrm{InnAut}(G)$.

Next we state three propositions:

(20)[1] For all elements x, y of $\mathrm{InnAutGroup}(G)$ and for all elements f, g of $\mathrm{InnAut}(G)$ such that $x = f$ and $y = g$ holds $x \cdot y = f \cdot g$.

(21) $\mathrm{id}_{(\text{the carrier of } G)} = 1_{\mathrm{InnAutGroup}(G)}$.

(22) For every element f of $\mathrm{InnAut}(G)$ and for every element g of $\mathrm{InnAutGroup}(G)$ such that $f = g$ holds $f^{-1} = g^{-1}$.

Let us consider G, b. The functor $\mathrm{Conjugate}(b)$ yields an element of $\mathrm{InnAut}(G)$ and is defined by:

(Def.6) For every a holds $(\mathrm{Conjugate}(b))(a) = a^b$.

The following propositions are true:

(23) For all a, b holds $\mathrm{Conjugate}(a \cdot b) = \mathrm{Conjugate}(b) \cdot \mathrm{Conjugate}(a)$.

(24) $\mathrm{Conjugate}(1_G) = \mathrm{id}_{(\text{the carrier of } G)}$.

(25) For every a holds $(\mathrm{Conjugate}(1_G))(a) = a$.

(26) For every a holds $\mathrm{Conjugate}(a) \cdot \mathrm{Conjugate}(a^{-1}) = \mathrm{Conjugate}(1_G)$.

(27) For every a holds $\mathrm{Conjugate}(a^{-1}) \cdot \mathrm{Conjugate}(a) = \mathrm{Conjugate}(1_G)$.

(28) For every a holds $\mathrm{Conjugate}(a^{-1}) = (\mathrm{Conjugate}(a))^{-1}$.

[1] The proposition (19) has been removed.

(29) For every a holds $\mathrm{Conjugate}(a) \cdot \mathrm{Conjugate}(1_G) = \mathrm{Conjugate}(a)$ and $\mathrm{Conjugate}(1_G) \cdot \mathrm{Conjugate}(a) = \mathrm{Conjugate}(a)$.

(30) For every element f of $\mathrm{InnAut}(G)$ holds $f \cdot \mathrm{Conjugate}(1_G) = f$ and $\mathrm{Conjugate}(1_G) \cdot f = f$.

(31) For every G holds $\mathrm{InnAutGroup}(G)$ and ${}^{G}/_{\mathrm{Z}(G)}$ are isomorphic.

(32) For every G such that G is a commutative group and for every element f of $\mathrm{InnAutGroup}(G)$ holds $f = 1_{\mathrm{InnAutGroup}(G)}$.

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