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Binary Arithmetics, Addition and Subtraction of Integers

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Summary. This article is a continuation of [6] and presents the concepts of binary arithmetic operations for integers. There is introduced 2's complement representation of integers and natural numbers to integers are expanded. The binary addition and subtraction for integers are defined and theorems on the relationship between binary and numerical operations presented.

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The notation and terminology used here are introduced in the following papers: [8], [5], [4], [9], [11], [7], [2], [1], [3], [10], and [6].

Let X be a non empty set, let D be a non empty subset of X, let x, y be arbitrary, and let a, b be elements of D. Then $(x = y \rightarrow a, b)$ is an element of D.

We follow the rules: i will be a natural number, n will be a non empty natural number, and x, y, z_1, z_2 will be tuples of n and *Boolean*.

Let us consider n. The functor $\mathrm{Bin1}(n)$ yielding a tuple of n and *Boolean* is defined by:

(Def.1) For every i such that $i \in \mathrm{Seg}\, n$ holds $\pi_i \,\mathrm{Bin1}(n) = (i = 1 \rightarrow \mathit{true}, \mathit{false})$.

Let us consider n, x. The functor $\mathrm{Neg2}(x)$ yielding a tuple of n and *Boolean* is defined by:

(Def.2) $\mathrm{Neg2}(x) = \neg x + \mathrm{Bin1}(n)$.

Let us consider n, x. The functor $\mathrm{Intval}(x)$ yielding an integer is defined by:

(Def.3) (i) $\mathrm{Intval}(x) = \mathrm{Absval}(x)$ if $\pi_n x = \mathit{false}$,

(ii) $\mathrm{Intval}(x) = \mathrm{Absval}(x) - (\text{the } n\text{-th power of } 2)$, otherwise.

Let us consider n, z_1, z_2. The functor $\mathrm{Int\_add\_ovfl}(z_1, z_2)$ yields an element of *Boolean* and is defined by:

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(Def.4) $\mathrm{Int\_add\_ovfl}(z_1, z_2) = \neg\pi_n z_1 \wedge \neg\pi_n z_2 \wedge \pi_n \mathrm{carry}(z_1, z_2)$.

Let us consider n, z_1, z_2. The functor $\mathrm{Int\_add\_udfl}(z_1, z_2)$ yields an element of *Boolean* and is defined by:

(Def.5) $\mathrm{Int\_add\_udfl}(z_1, z_2) = \pi_n z_1 \wedge \pi_n z_2 \wedge \neg\pi_n \mathrm{carry}(z_1, z_2)$.

The following propositions are true:

(1) For every tuple z_1 of 1 and *Boolean* such that $z_1 = \langle \mathit{false} \rangle$ holds $\mathrm{Absval}(z_1) = 0$.

(2) For every tuple z_1 of 1 and *Boolean* such that $z_1 = \langle \mathit{true} \rangle$ holds $\mathrm{Absval}(z_1) = 1$.

(3) For every tuple z_1 of 2 and *Boolean* such that $z_1 = \langle \mathit{false} \rangle \frown \langle \mathit{false} \rangle$ holds $\mathrm{Intval}(z_1) = 0$.

(4) For every tuple z_1 of 2 and *Boolean* such that $z_1 = \langle \mathit{true} \rangle \frown \langle \mathit{false} \rangle$ holds $\mathrm{Intval}(z_1) = 1$.

(5) For every tuple z_1 of 2 and *Boolean* such that $z_1 = \langle \mathit{false} \rangle \frown \langle \mathit{true} \rangle$ holds $\mathrm{Intval}(z_1) = -2$.

(6) For every tuple z_1 of 2 and *Boolean* such that $z_1 = \langle \mathit{true} \rangle \frown \langle \mathit{true} \rangle$ holds $\mathrm{Intval}(z_1) = -1$.

(7) For every i such that $i \in \mathrm{Seg}\, n$ and $i = 1$ holds $\pi_i \, \mathrm{Bin1}(n) = \mathit{true}$.

(8) For every i such that $i \in \mathrm{Seg}\, n$ and $i \neq 1$ holds $\pi_i \, \mathrm{Bin1}(n) = \mathit{false}$.

(9) For every n holds $\mathrm{Bin1}(n+1) = (\mathrm{Bin1}(n)) \frown \langle \mathit{false} \rangle$.

(10) For every n holds $\mathrm{Intval}((\mathrm{Bin1}(n)) \frown \langle \mathit{false} \rangle) = 1$.

(11) For every n and for every tuple z of n and *Boolean* and for every element d of *Boolean* holds $\neg(z \frown \langle d \rangle) = (\neg z) \frown \langle \neg d \rangle$.

(12) Given n, and let z be a tuple of n and *Boolean*, and let d be an element of *Boolean*. Then $\mathrm{Intval}(z \frown \langle d \rangle) = \mathrm{Absval}(z) - ((d = \mathit{false} \rightarrow 0, \text{the } n\text{-th power of } 2)$ **qua** natural number).

(13) Given n, and let z_1, z_2 be tuples of n and *Boolean*, and let d_1, d_2 be elements of *Boolean*. Then $(\mathrm{Intval}(z_1 \frown \langle d_1 \rangle + z_2 \frown \langle d_2 \rangle) + (\mathrm{Int\_add\_ovfl}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle) = \mathit{false} \rightarrow 0, \text{the } n+1\text{-th power of } 2)) - (\mathrm{Int\_add\_udfl}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle) = \mathit{false} \rightarrow 0, \text{the } n+1\text{-th power of } 2) = \mathrm{Intval}(z_1 \frown \langle d_1 \rangle) + \mathrm{Intval}(z_2 \frown \langle d_2 \rangle)$.

(14) Given n, and let z_1, z_2 be tuples of n and *Boolean*, and let d_1, d_2 be elements of *Boolean*. Then $\mathrm{Intval}(z_1 \frown \langle d_1 \rangle + z_2 \frown \langle d_2 \rangle) = ((\mathrm{Intval}(z_1 \frown \langle d_1 \rangle) + \mathrm{Intval}(z_2 \frown \langle d_2 \rangle)) - (\mathrm{Int\_add\_ovfl}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle) = \mathit{false} \rightarrow 0, \text{the } n+1\text{-th power of } 2)) + (\mathrm{Int\_add\_udfl}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle) = \mathit{false} \rightarrow 0, \text{the } n+1\text{-th power of } 2)$.

(15) For every n and for every tuple x of n and *Boolean* holds $\mathrm{Absval}(\neg x) = (-\mathrm{Absval}(x) + (\text{the } n\text{-th power of } 2)) - 1$.

(16) For every n and for every tuple z of n and *Boolean* and for every element d of *Boolean* holds $\mathrm{Neg2}(z \frown \langle d \rangle) = (\mathrm{Neg2}(z)) \frown \langle \neg d \oplus \mathrm{add\_ovfl}(\neg z, \mathrm{Bin1}(n)) \rangle$.

(17) Given n, and let z be a tuple of n and *Boolean*, and let d be an element of *Boolean*. Then $\mathrm{Intval}(\mathrm{Neg2}(z \frown \langle d \rangle)) + (\mathrm{Int\_add\_ovfl}(\neg(z \frown \langle d \rangle), \mathrm{Bin1}(n+1)) = \mathit{false} \rightarrow 0, \text{the } n+1\text{-th power of } 2) = -\mathrm{Intval}(z \frown \langle d \rangle)$.

(18) For every n and for every tuple z of n and *Boolean* and for every element d of *Boolean* holds $\mathrm{Neg2}(\mathrm{Neg2}(z \frown \langle d \rangle)) = z \frown \langle d \rangle$.

Let us consider n, x, y. The functor $x - y$ yielding a tuple of n and *Boolean* is defined as follows:

(Def.6) For every i such that $i \in \mathrm{Seg}\, n$ holds $\pi_i(x - y) = \pi_i x \oplus \pi_i \mathrm{Neg2}(y) \oplus \pi_i \mathrm{carry}(x, \mathrm{Neg2}(y))$.

One can prove the following three propositions:

(19) For every n and for all tuples x, y of n and *Boolean* holds $x - y = x + \mathrm{Neg2}(y)$.

(20) For every n and for all tuples z_1, z_2 of n and *Boolean* and for all elements d_1, d_2 of *Boolean* holds $z_1 \frown \langle d_1 \rangle - z_2 \frown \langle d_2 \rangle = (z_1 + \mathrm{Neg2}(z_2)) \frown \langle d_1 \oplus \neg d_2 \oplus \mathrm{add\_ovfl}(\neg z_2, \mathrm{Bin1}(n)) \oplus \mathrm{add\_ovfl}(z_1, \mathrm{Neg2}(z_2)) \rangle$.

(21) Given n, and let z_1, z_2 be tuples of n and *Boolean*, and let d_1, d_2 be elements of *Boolean*. Then $((\mathrm{Intval}(z_1 \frown \langle d_1 \rangle - z_2 \frown \langle d_2 \rangle) + (\mathrm{Int\_add\_ovfl}(z_1 \frown \langle d_1 \rangle, \mathrm{Neg2}(z_2 \frown \langle d_2 \rangle)) = \mathit{false} \rightarrow 0, \text{the } n+1\text{-th power of } 2)) - (\mathrm{Int\_add\_udfl}(z_1 \frown \langle d_1 \rangle, \mathrm{Neg2}(z_2 \frown \langle d_2 \rangle)) = \mathit{false} \rightarrow 0, \text{the } n+1\text{-th power of } 2)) + (\mathrm{Int\_add\_ovfl}(\neg(z_2 \frown \langle d_2 \rangle), \mathrm{Bin1}(n+1)) = \mathit{false} \rightarrow 0, \text{the } n+1\text{-th power of } 2) = \mathrm{Intval}(z_1 \frown \langle d_1 \rangle) - \mathrm{Intval}(z_2 \frown \langle d_2 \rangle)$.

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