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Boolean Properties of Lattices

Agnieszka Julia Marasik
Warsaw University
Białystok

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The article [1] provides the terminology and notation for this paper.

1. General lattice

We follow the rules: L will be a lattice and X, Y, Z, V will be elements of the carrier of L.

Let us consider L, X, Y. The functor $X \setminus Y$ yielding an element of the carrier of L is defined by:

(Def.1) $X \setminus Y = X \sqcap Y^{\rm c}$.

Let us consider L, X, Y. The functor $X \dot{-} Y$ yields an element of the carrier of L and is defined by:

(Def.2) $X \dot{-} Y = (X \setminus Y) \sqcup (Y \setminus X)$.

Let us consider L, X, Y. Let us observe that $X = Y$ if and only if:

(Def.3) $X \sqsubseteq Y$ and $Y \sqsubseteq X$.

Let us consider L, X, Y. We say that X meets Y if and only if:

(Def.4) $X \sqcap Y \neq \perp_L$.

We introduce X misses Y as an antonym of X meets Y.

We now state a number of propositions:

(1) $X \sqsubseteq X \sqcup Y$ and $Y \sqsubseteq X \sqcup Y$.

(3)[1] If $X \sqcup Y \sqsubseteq Z$, then $X \sqsubseteq Z$ and $Y \sqsubseteq Z$.

(4) $X \sqcap Y \sqsubseteq X \sqcup Z$.

(5) If $X \sqsubseteq Y$, then $X \sqcap Z \sqsubseteq Y \sqcap Z$ and $Z \sqcap X \sqsubseteq Z \sqcap Y$.

(6) If $X \sqsubseteq Z$, then $X \setminus Y \sqsubseteq Z$.

[1]The proposition (2) has been removed.

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(7) If $X \sqsubseteq Y$, then $X \setminus Z \sqsubseteq Y \setminus Z$.

(8) $X \setminus Y \sqsubseteq X$.

(9) $X \setminus Y \sqsubseteq X \dot{-} Y$.

(10) If $X \setminus Y \sqsubseteq Z$ and $Y \setminus X \sqsubseteq Z$, then $X \dot{-} Y \sqsubseteq Z$.

(11) $X = Y \sqcup Z$ iff $Y \sqsubseteq X$ and $Z \sqsubseteq X$ and for every V such that $Y \sqsubseteq V$ and $Z \sqsubseteq V$ holds $X \sqsubseteq V$.

(12) $X = Y \sqcap Z$ iff $X \sqsubseteq Y$ and $X \sqsubseteq Z$ and for every V such that $V \sqsubseteq Y$ and $V \sqsubseteq Z$ holds $V \sqsubseteq X$.

(13) If $X \sqcup Y = Y$ or $Y \sqcup X = Y$, then $X \sqsubseteq Y$.

(14) $X \sqcap (Y \setminus Z) = X \sqcap Y \setminus Z$.

(15) If X meets Y, then Y meets X.

(16) X meets X iff $X \neq \bot_L$.

(17) $X \dot{-} Y = Y \dot{-} X$.

2. Modular lattice

In the sequel L will denote a modular lattice and X, Y will denote elements of the carrier of L.

The following three propositions are true:

(18) If $Y \sqsubseteq X$ and $X \sqcap Y = \bot_L$, then $Y = \bot_L$.

(20)[2] If $X \sqsubseteq Y$, then $X \sqcup Y = Y$ and $Y \sqcup X = Y$.

(21) If X misses Y, then Y misses X.

3. Distributive lattice

In the sequel L will denote a distributive lattice and X, Y, Z will denote elements of the carrier of L.

Next we state three propositions:

(22) If $X \sqcap Y \sqcup X \sqcap Z = X$, then $X \sqsubseteq Y \sqcup Z$.

(23) $X \sqcap Y \sqcup Y \sqcap Z \sqcup Z \sqcap X = (X \sqcup Y) \sqcap (Y \sqcup Z) \sqcap (Z \sqcup X)$.

(24) $(X \sqcup Y) \setminus Z = (X \setminus Z) \sqcup (Y \setminus Z)$.

[2] The proposition (19) has been removed.

4. Distributive lower bounded lattice

In the sequel L will denote a lower bound lattice and X, Y, Z will denote elements of the carrier of L.

The following propositions are true:

(25) If $X \sqsubseteq \perp_L$, then $X = \perp_L$.

(26) If $X \sqsubseteq Y$ and $X \sqsubseteq Z$ and $Y \sqcap Z = \perp_L$, then $X = \perp_L$.

(27) $X \sqcup Y = \perp_L$ iff $X = \perp_L$ and $Y = \perp_L$.

(28) If $X \sqsubseteq Y$ and $Y \sqcap Z = \perp_L$, then $X \sqcap Z = \perp_L$.

(29) $\perp_L \setminus X = \perp_L$.

(30) If X meets Y and $Y \sqsubseteq Z$, then X meets Z.

(31) If X meets $Y \sqcap Z$, then X meets Y and X meets Z.

(32) If X meets $Y \setminus Z$, then X meets Y.

(33) X misses $\perp_L$.

(34) If X misses Z and $Y \sqsubseteq Z$, then X misses Y.

(35) If X misses Y or X misses Z, then X misses $Y \sqcap Z$.

(36) If $X \sqsubseteq Y$ and $X \sqsubseteq Z$ and Y misses Z, then $X = \perp_L$.

(37) If X misses Y, then $Z \sqcap X$ misses $Z \sqcap Y$ and $X \sqcap Z$ misses $Y \sqcap Z$.

5. Boolean lattice

We follow a convention: L will be a Boolean lattice and X, Y, Z, V will be elements of the carrier of L.

Next we state a number of propositions:

(38) If $X \setminus Y \sqsubseteq Z$, then $X \sqsubseteq Y \sqcup Z$.

(39) If $X \sqsubseteq Y$, then $Z \setminus Y \sqsubseteq Z \setminus X$.

(40) If $X \sqsubseteq Y$ and $Z \sqsubseteq V$, then $X \setminus V \sqsubseteq Y \setminus Z$.

(41) If $X \sqsubseteq Y \sqcup Z$, then $X \setminus Y \sqsubseteq Z$ and $X \setminus Z \sqsubseteq Y$.

(42) $X^{\rm c} \sqsubseteq (X \sqcap Y)^{\rm c}$ and $Y^{\rm c} \sqsubseteq (X \sqcap Y)^{\rm c}$.

(43) $(X \sqcup Y)^{\rm c} \sqsubseteq X^{\rm c}$ and $(X \sqcup Y)^{\rm c} \sqsubseteq Y^{\rm c}$.

(44) If $X \sqsubseteq Y \setminus X$, then $X = \perp_L$.

(45) If $X \sqsubseteq Y$, then $Y = X \sqcup (Y \setminus X)$ and $Y = (Y \setminus X) \sqcup X$.

(46) $X \setminus Y = \perp_L$ iff $X \sqsubseteq Y$.

(47) If $X \sqsubseteq Y \sqcup Z$ and $X \sqcap Z = \perp_L$, then $X \sqsubseteq Y$.

(48) $X \sqcup Y = (X \setminus Y) \sqcup Y$.

(49) $X \setminus (X \sqcup Y) = \perp_L$ and $X \setminus (Y \sqcup X) = \perp_L$.

(50) $X \setminus X \sqcap Y = X \setminus Y$ and $X \setminus Y \sqcap X = X \setminus Y$.

(51) $(X \setminus Y) \sqcap Y = \perp_L$ and $Y \sqcap (X \setminus Y) = \perp_L$.

(52) $X \sqcup (Y \setminus X) = X \sqcup Y$ and $(Y \setminus X) \sqcup X = Y \sqcup X$.
(53) $X \sqcap Y \sqcup (X \setminus Y) = X$ and $(X \setminus Y) \sqcup X \sqcap Y = X$.
(54) $X \setminus (Y \setminus Z) = (X \setminus Y) \sqcup X \sqcap Z$.
(55) $X \setminus (X \setminus Y) = X \sqcap Y$.
(56) $(X \sqcup Y) \setminus Y = X \setminus Y$.
(57) $X \sqcap Y = \perp_L$ iff $X \setminus Y = X$.
(58) $X \setminus (Y \sqcup Z) = (X \setminus Y) \sqcap (X \setminus Z)$.
(59) $X \setminus Y \sqcap Z = (X \setminus Y) \sqcup (X \setminus Z)$.
(60) $X \sqcap (Y \setminus Z) = X \sqcap Y \setminus X \sqcap Z$ and $(Y \setminus Z) \sqcap X = Y \sqcap X \setminus Z \sqcap X$.
(61) $(X \sqcup Y) \setminus X \sqcap Y = (X \setminus Y) \sqcup (Y \setminus X)$.
(62) $X \setminus Y \setminus Z = X \setminus (Y \sqcup Z)$.
(63) If $X \setminus Y = Y \setminus X$, then $X = Y$.
(64) $(\perp_L)^{\mathrm{c}} = \top_L$.
(65) $(\top_L)^{\mathrm{c}} = \perp_L$.
(66) $X \setminus X = \perp_L$.
(67) $X \setminus \perp_L = X$.
(68) $(X \setminus Y)^{\mathrm{c}} = X^{\mathrm{c}} \sqcup Y$.
(69) X meets $Y \sqcup Z$ iff X meets Y or X meets Z.
(70) $X \sqcap Y$ misses $X \setminus Y$.
(71) X misses $Y \sqcup Z$ iff X misses Y and X misses Z.
(72) $X \setminus Y$ misses Y.
(73) If X misses Y, then $(X \sqcup Y) \setminus Y = X$ and $(X \sqcup Y) \setminus X = Y$.
(74) If $X^{\mathrm{c}} \sqcup Y^{\mathrm{c}} = X \sqcup Y$ and X misses X^{c} and Y misses Y^{c}, then $X = Y^{\mathrm{c}}$ and $Y = X^{\mathrm{c}}$.
(75) If $X^{\mathrm{c}} \sqcup Y^{\mathrm{c}} = X \sqcup Y$ and Y misses X^{c} and X misses Y^{c}, then $X = X^{\mathrm{c}}$ and $Y = Y^{\mathrm{c}}$.
(76) $X \dot{-} \perp_L = X$ and $\perp_L \dot{-} X = X$.
(77) $X \dot{-} X = \perp_L$.
(78) $X \sqcap Y$ misses $X \dot{-} Y$.
(79) $X \sqcup Y = X \dot{-} (Y \setminus X)$.
(80) $X \dot{-} X \sqcap Y = X \setminus Y$.
(81) $X \sqcup Y = (X \dot{-} Y) \sqcup X \sqcap Y$.
(82) $X \dot{-} Y \dot{-} X \sqcap Y = X \sqcup Y$.
(83) $X \dot{-} Y \dot{-} (X \sqcup Y) = X \sqcap Y$.
(84) $X \dot{-} Y = (X \sqcup Y) \setminus X \sqcap Y$.
(85) $(X \dot{-} Y) \setminus Z = (X \setminus (Y \sqcup Z)) \sqcup (Y \setminus (X \sqcup Z))$.
(86) $X \setminus (Y \dot{-} Z) = (X \setminus (Y \sqcup Z)) \sqcup X \sqcap Y \sqcap Z$.
(87) $(X \dot{-} Y) \dot{-} Z = X \dot{-} (Y \dot{-} Z)$.
(88) $(X \dot{-} Y)^{\mathrm{c}} = X \sqcap Y \sqcup X^{\mathrm{c}} \sqcap Y^{\mathrm{c}}$.

References

[1] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

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