

Solvable Groups

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Summary. The concept of solvable group is introduced. Some theorems concerning heirdom of solvability are proved.

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The articles [7], [13], [3], [4], [11], [6], [5], [2], [1], [9], [10], [8], and [12] provide the terminology and notation for this paper.

In this paper G denotes a group and i denotes a natural number.

A group is solvable if it satisfies the condition (Def.1).

(Def.1) There exists a finite sequence F of elements of SubGr it such that

- (i) $\text{len } F > 0$,
- (ii) $F(1) = \Omega_{it}$,
- (iii) $F(\text{len } F) = \{\mathbf{1}\}_{it}$, and
- (iv) for every i such that $i \in \text{dom } F$ and $i + 1 \in \text{dom } F$ and for all strict subgroups G_1, G_2 of it such that $G_1 = F(i)$ and $G_2 = F(i + 1)$ holds G_2 is a strict normal subgroup of G_1 and for every normal subgroup N of G_1 such that $N = G_2$ holds G_1/N is commutative.

One can check that there exists a group which is solvable and strict.

One can prove the following propositions:

- (1) Let G be a strict group and let H, F_1, F_2 be strict subgroups of G . Suppose F_1 is a normal subgroup of F_2 . Then $F_1 \cap H$ is a normal subgroup of $F_2 \cap H$.
- (2) Let G be a strict group, and let F_2 be a strict subgroup of G , and let F_1 be a strict normal subgroup of F_2 , and let a, b be elements of F_2 . Then $a \cdot F_1 \cdot (b \cdot F_1) = (a \cdot b) \cdot F_1$.
- (3) Let G be a strict group, and let H, F_2 be strict subgroups of G , and let F_1 be a strict normal subgroup of F_2 , and let G_2 be a strict subgroup of G . Suppose $G_2 = H \cap F_2$. Let G_1 be a normal subgroup of G_2 . Suppose

$G_1 = H \cap F_1$. Then there exists a subgroup G_3 of F_2/F_1 such that G_2/G_1 and G_3 are isomorphic.

- (4) Let G be a strict group, and let H, F_2 be strict subgroups of G , and let F_1 be a strict normal subgroup of F_2 , and let G_2 be a strict subgroup of G . Suppose $G_2 = F_2 \cap H$. Let G_1 be a normal subgroup of G_2 . Suppose $G_1 = F_1 \cap H$. Then there exists a subgroup G_3 of F_2/F_1 such that G_2/G_1 and G_3 are isomorphic.
- (5) For every solvable strict group G holds every strict subgroup of G is solvable.
- (6) Let G be a strict group. Given a finite sequence F of elements of $\text{SubGr } G$ such that
- (i) $\text{len } F > 0$,
 - (ii) $F(1) = \Omega_G$,
 - (iii) $F(\text{len } F) = \{\mathbf{1}\}_G$, and
 - (iv) for every i such that $i \in \text{dom } F$ and $i + 1 \in \text{dom } F$ and for all strict subgroups G_1, G_2 of G such that $G_1 = F(i)$ and $G_2 = F(i + 1)$ holds G_2 is a strict normal subgroup of G_1 and for every normal subgroup N of G_1 such that $N = G_2$ holds G_1/N is a cyclic group.
- Then G is solvable.
- (7) Every strict commutative group is strict and solvable.

Let G, H be strict groups, let g be a homomorphism from G to H , and let A be a subgroup of G . The functor $g \upharpoonright A$ yielding a homomorphism from A to H is defined as follows:

(Def.2) $g \upharpoonright A = g \upharpoonright (\text{the carrier of } A)$.

Let G, H be strict groups, let g be a homomorphism from G to H , and let A be a subgroup of G . The functor $g^\circ A$ yields a strict subgroup of H and is defined as follows:

(Def.3) $g^\circ A = \text{Im}(g \upharpoonright A)$.

Next we state a number of propositions:

- (8) Let G, H be strict groups, and let g be a homomorphism from G to H , and let A be a subgroup of G . Then $\text{rng}(g \upharpoonright A) = g^\circ(\text{the carrier of } A)$.
- (9) Let G, H be strict groups, and let g be a homomorphism from G to H , and let A be a strict subgroup of G . Then the carrier of $g^\circ A = g^\circ(\text{the carrier of } A)$.
- (10) Let G, H be strict groups, and let h be a homomorphism from G to H , and let A be a strict subgroup of G . Then $\text{Im}(h \upharpoonright A)$ is a strict subgroup of $\text{Im } h$.
- (11) Let G, H be strict groups, and let h be a homomorphism from G to H , and let A be a strict subgroup of G . Then $h^\circ A$ is a strict subgroup of $\text{Im } h$.
- (12) For all strict groups G, H and for every homomorphism h from G to H holds $h^\circ(\{\mathbf{1}\}_G) = \{\mathbf{1}\}_H$ and $h^\circ(\Omega_G) = \Omega_{\text{Im } h}$.

- (13) Let G, H be strict groups, and let h be a homomorphism from G to H , and let A, B be strict subgroups of G . If A is a subgroup of B , then $h^\circ A$ is a subgroup of $h^\circ B$.
- (14) Let G, H be strict groups, and let h be a homomorphism from G to H , and let A be a strict subgroup of G , and let a be an element of G . Then $h(a) \cdot h^\circ A = h^\circ(a \cdot A)$ and $h^\circ A \cdot h(a) = h^\circ(A \cdot a)$.
- (15) Let G, H be strict groups, and let h be a homomorphism from G to H , and let A, B be subsets of G . Then $h^\circ A \cdot h^\circ B = h^\circ(A \cdot B)$.
- (16) Let G, H be strict groups, and let h be a homomorphism from G to H , and let A, B be strict subgroups of G . Suppose A is a strict normal subgroup of B . Then $h^\circ A$ is a strict normal subgroup of $h^\circ B$.
- (17) Let G, H be strict groups and let h be a homomorphism from G to H . If G is a solvable group, then $\text{Im } h$ is solvable.

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