

Free Many Sorted Universal Algebra

Beata Perkowska
Warsaw University
Białystok

MML Identifier: MSAFREE.

The terminology and notation used in this paper are introduced in the following papers: [21], [24], [25], [11], [22], [12], [7], [18], [13], [10], [2], [4], [5], [23], [14], [6], [1], [16], [3], [8], [20], [17], [19], [9], and [15].

1. PRELIMINARIES

The following proposition is true

- (1) Let I be a set, and let J be a non empty set, and let f be a function from I into J^* , and let X be a many sorted set of J , and let p be an element of J^* , and let x be arbitrary. If $x \in I$ and $p = f(x)$, then $(X^\# \cdot f)(x) = \prod(X \cdot p)$.

Let I be a set, let A, B be many sorted sets of I , let C be a many sorted subset of A , and let F be a many sorted function from A into B . The functor $F \upharpoonright C$ yielding a many sorted function from C into B is defined as follows:

- (Def.1) For arbitrary i such that $i \in I$ and for every function f from $A(i)$ into $B(i)$ such that $f = F(i)$ holds $(F \upharpoonright C)(i) = f \upharpoonright C(i)$.

Let I be a set, let X be a many sorted set of I , and let i be arbitrary. Let us assume that $i \in I$. The functor $\text{coprod}(i, X)$ yields a set and is defined as follows:

- (Def.2) For arbitrary x holds $x \in \text{coprod}(i, X)$ iff there exists arbitrary a such that $a \in X(i)$ and $x = \langle a, i \rangle$.

Let I be a set and let X be a many sorted set of I . Then disjoint X is a many sorted set of I and it can be characterized by the condition:

- (Def.3) For arbitrary i such that $i \in I$ holds $(\text{disjoint } X)(i) = \text{coprod}(i, X)$.

We introduce $\text{coprod}(X)$ as a synonym of disjoint X .

Let I be a non empty set and let X be a non-empty many sorted set of I . One can verify that $\text{coprod}(X)$ is non-empty.

Let I be a non empty set and let X be a non-empty many sorted set of I . One can check that $\bigcup X$ is non empty.

We now state the proposition

- (2) Let I be a set, and let X be a many sorted set of I , and let i be arbitrary. If $i \in I$, then $X(i) \neq \emptyset$ iff $(\text{coprod}(X))(i) \neq \emptyset$.

2. FREE MANY SORTED UNIVERSAL ALGEBRA - GENERAL NOTIONS

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S . A subset of U_0 is said to be a generator set of U_0 if:

(Def.4) The sorts of $\text{Gen}(it) =$ the sorts of U_0 .

Next we state the proposition

- (3) Let S be a non void non empty many sorted signature, and let U_0 be a strict non-empty algebra over S , and let A be a subset of U_0 . Then A is a generator set of U_0 if and only if $\text{Gen}(A) = U_0$.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S . A generator set of U_0 is free if it satisfies the condition (Def.5).

(Def.5) Let U_1 be a non-empty algebra over S and let f be a many sorted function from it into the sorts of U_1 . Then there exists a many sorted function h from U_0 into U_1 such that h is a homomorphism of U_0 into U_1 and $h \upharpoonright it = f$.

Let S be a non void non empty many sorted signature. A non-empty algebra over S is free if:

(Def.6) There exists generator set of it which is free.

The following proposition is true

- (4) Let S be a non void non empty many sorted signature and let X be a many sorted set of the carrier of S . Then $\bigcup \text{coprod}(X) \cap \{ \text{the operation symbols of } S, \{ \text{the carrier of } S \} \} = \emptyset$.

3. SEMIDISJOINT MANY SORTED SIGNATURE

Let S be a non void many sorted signature. Note that the operation symbols of S is non empty.

Let S be a non void non empty many sorted signature and let X be a many sorted set of the carrier of S . The functor $\text{REL}(X)$ yields a relation between $\{ \text{the operation symbols of } S, \{ \text{the carrier of } S \} \} \cup \bigcup \text{coprod}(X)$ and

($\{$ the operation symbols of S , $\{$ the carrier of S $\} \cup \bigcup \text{coprod}(X)$) $\}^*$ and is defined by the condition (Def.9).

(Def.9)¹ Let a be an element of ($\{$ the operation symbols of S , $\{$ the carrier of S $\} \cup \bigcup \text{coprod}(X)$) and let b be an element of ($\{$ the operation symbols of S , $\{$ the carrier of S $\} \cup \bigcup \text{coprod}(X)$) $\}^*$. Then $\langle a, b \rangle \in \text{REL}(X)$ if and only if the following conditions are satisfied:

- (i) $a \in \{$ the operation symbols of S , $\{$ the carrier of S $\}$, and
- (ii) for every operation symbol o of S such that $\langle o, \text{the carrier of } S \rangle = a$ holds $\text{len } b = \text{len Arity}(o)$ and for arbitrary x such that $x \in \text{dom } b$ holds if $b(x) \in \{$ the operation symbols of S , $\{$ the carrier of S $\}$, then for every operation symbol o_1 of S such that $\langle o_1, \text{the carrier of } S \rangle = b(x)$ holds the result sort of $o_1 = \text{Arity}(o)(x)$ and if $b(x) \in \bigcup \text{coprod}(X)$, then $b(x) \in \text{coprod}(\text{Arity}(o)(x), X)$.

In the sequel S will be a non void non empty many sorted signature, X will be a many sorted set of the carrier of S , o will be an operation symbol of S , and b will be an element of ($\{$ the operation symbols of S , $\{$ the carrier of S $\} \cup \bigcup \text{coprod}(X)$) $\}^*$.

Next we state the proposition

(5) $\langle \langle o, \text{the carrier of } S \rangle, b \rangle \in \text{REL}(X)$ if and only if the following conditions are satisfied:

- (i) $\text{len } b = \text{len Arity}(o)$, and
- (ii) for arbitrary x such that $x \in \text{dom } b$ holds if $b(x) \in \{$ the operation symbols of S , $\{$ the carrier of S $\}$, then for every operation symbol o_1 of S such that $\langle o_1, \text{the carrier of } S \rangle = b(x)$ holds the result sort of $o_1 = \text{Arity}(o)(x)$ and if $b(x) \in \bigcup \text{coprod}(X)$, then $b(x) \in \text{coprod}(\text{Arity}(o)(x), X)$.

Let S be a non void non empty many sorted signature and let X be a many sorted set of the carrier of S . The functor $\text{DTConMSA}(X)$ yielding a strict tree construction structure is defined as follows:

(Def.10) $\text{DTConMSA}(X) = \langle \{$ the operation symbols of S , $\{$ the carrier of S $\} \cup \bigcup \text{coprod}(X), \text{REL}(X) \rangle$.

Let S be a non void non empty many sorted signature and let X be a many sorted set of the carrier of S . Observe that $\text{DTConMSA}(X)$ is non empty.

We now state the proposition

(6) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S . Then the nonterminals of $\text{DTConMSA}(X) = \{$ the operation symbols of S , $\{$ the carrier of S $\}$ and the terminals of $\text{DTConMSA}(X) = \bigcup \text{coprod}(X)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S . Observe that $\text{DTConMSA}(X)$ has terminals, nonterminals, and useful nonterminals.

One can prove the following proposition

¹The definitions (Def.7) and (Def.8) have been removed.

- (7) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S , and let t be arbitrary. Then $t \in$ the terminals of $\text{DTConMSA}(X)$ if and only if there exists a sort symbol s of S and there exists arbitrary x such that $x \in X(s)$ and $t = \langle x, s \rangle$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S , and let o be an operation symbol of S . The functor $\text{Sym}(o, X)$ yielding a symbol of $\text{DTConMSA}(X)$ is defined by:

(Def.11) $\text{Sym}(o, X) = \langle o, \text{the carrier of } S \rangle$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S , and let s be a sort symbol of S . The functor $\text{FreeSort}(X, s)$ yielding a non empty subset of $\text{TS}(\text{DTConMSA}(X))$ is defined by the condition (Def.12).

(Def.12) $\text{FreeSort}(X, s) = \{a : a \text{ ranges over elements of } \text{TS}(\text{DTConMSA}(X)), \bigvee_x x \in X(s) \wedge a = \text{the root tree of } \langle x, s \rangle \vee \bigvee_o \langle o, \text{the carrier of } S \rangle = a(\varepsilon) \wedge \text{the result sort of } o = s\}$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S . The functor $\text{FreeSorts}(X)$ yielding a non-empty many sorted set of the carrier of S is defined by:

(Def.13) For every sort symbol s of S holds $(\text{FreeSorts}(X))(s) = \text{FreeSort}(X, s)$.

The following propositions are true:

- (8) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S , and let o be an operation symbol of S , and let x be arbitrary. Suppose $x \in ((\text{FreeSorts}(X))^{\#} \cdot (\text{the arity of } S))(o)$. Then x is a finite sequence of elements of $\text{TS}(\text{DTConMSA}(X))$.
- (9) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S , and let o be an operation symbol of S , and let p be a finite sequence of elements of $\text{TS}(\text{DTConMSA}(X))$. Then $p \in ((\text{FreeSorts}(X))^{\#} \cdot (\text{the arity of } S))(o)$ if and only if $\text{dom } p = \text{dom Arity}(o)$ and for every natural number n such that $n \in \text{dom } p$ holds $p(n) \in \text{FreeSort}(X, \pi_n \text{ Arity}(o))$.
- (10) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S , and let o be an operation symbol of S , and let p be a finite sequence of elements of $\text{TS}(\text{DTConMSA}(X))$. Then $\text{Sym}(o, X) \Rightarrow$ the roots of p if and only if $p \in ((\text{FreeSorts}(X))^{\#} \cdot (\text{the arity of } S))(o)$.
- (11) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S , and let o be an operation symbol of S . Then $(\text{FreeSorts}(X) \cdot (\text{the result sort of } S))(o) \neq \emptyset$.
- (12) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S . Then $\bigcup \text{rng FreeSorts}(X) = \text{TS}(\text{DTConMSA}(X))$.

- (13) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S , and let s_1, s_2 be sort symbols of S . If $s_1 \neq s_2$, then $(\text{FreeSorts}(X))(s_1) \cap (\text{FreeSorts}(X))(s_2) = \emptyset$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S , and let o be an operation symbol of S . The functor $\text{DenOp}(o, X)$ yielding a function from $((\text{FreeSorts}(X))^{\#} \cdot (\text{the arity of } S))(o)$ into $(\text{FreeSorts}(X) \cdot (\text{the result sort of } S))(o)$ is defined by:

- (Def.14) For every finite sequence p of elements of $\text{TS}(\text{DTConMSA}(X))$ such that $\text{Sym}(o, X) \Rightarrow$ the roots of p holds $(\text{DenOp}(o, X))(p) = \text{Sym}(o, X)\text{-tree}(p)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S . The functor $\text{FreeOperations}(X)$ yielding a many sorted function from $(\text{FreeSorts}(X))^{\#} \cdot (\text{the arity of } S)$ into $\text{FreeSorts}(X) \cdot (\text{the result sort of } S)$ is defined as follows:

- (Def.15) For every operation symbol o of S holds $(\text{FreeOperations}(X))(o) = \text{DenOp}(o, X)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S . The functor $\text{Free}(X)$ yields a strict non-empty algebra over S and is defined by:

- (Def.16) $\text{Free}(X) = \langle \text{FreeSorts}(X), \text{FreeOperations}(X) \rangle$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S , and let s be a sort symbol of S . The functor $\text{FreeGenerator}(s, X)$ yields a non empty subset of $(\text{FreeSorts}(X))(s)$ and is defined as follows:

- (Def.17) For arbitrary x holds $x \in \text{FreeGenerator}(s, X)$ iff there exists arbitrary a such that $a \in X(s)$ and $x =$ the root tree of $\langle a, s \rangle$.

The following proposition is true

- (14) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set of the carrier of S , and let s be a sort symbol of S . Then $\text{FreeGenerator}(s, X) = \{\text{the root tree of } t: t \text{ ranges over symbols of } \text{DTConMSA}(X), t \in \text{the terminals of } \text{DTConMSA}(X) \wedge t_2 = s\}$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S . The functor $\text{FreeGenerator}(X)$ yielding a generator set of $\text{Free}(X)$ is defined as follows:

- (Def.18) For every sort symbol s of S holds $(\text{FreeGenerator}(X))(s) = \text{FreeGenerator}(s, X)$.

We now state two propositions:

- (15) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S . Then $\text{FreeGenerator}(X)$ is non-empty.
- (16) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S . Then

$\bigcup \text{rng FreeGenerator}(X) = \{\text{the root tree of } t: t \text{ ranges over symbols of DTConMSA}(X), t \in \text{the terminals of DTConMSA}(X)\}.$

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S , and let s be a sort symbol of S . The functor $\text{Reverse}(s, X)$ yielding a function from $\text{FreeGenerator}(s, X)$ into $X(s)$ is defined as follows:

(Def.19) For every symbol t of $\text{DTConMSA}(X)$ such that the root tree of $t \in \text{FreeGenerator}(s, X)$ holds $(\text{Reverse}(s, X))(\text{the root tree of } t) = t_1$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S . The functor $\text{Reverse}(X)$ yielding a many sorted function from $\text{FreeGenerator}(X)$ into X is defined by:

(Def.20) For every sort symbol s of S holds $(\text{Reverse}(X))(s) = \text{Reverse}(s, X)$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S , let A be a non-empty many sorted set of the carrier of S , let F be a many sorted function from $\text{FreeGenerator}(X)$ into A , and let t be a symbol of $\text{DTConMSA}(X)$. Let us assume that $t \in \text{the terminals of DTConMSA}(X)$. The functor $\pi(F, A, t)$ yielding an element of $\bigcup A$ is defined as follows:

(Def.21) For every function f such that $f = F(t_2)$ holds $\pi(F, A, t) = f(\text{the root tree of } t)$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set of the carrier of S , and let t be a symbol of $\text{DTConMSA}(X)$. Let us assume that there exists a finite sequence p such that $t \Rightarrow p$. The functor ${}^{\textcircled{a}}(X, t)$ yielding an operation symbol of S is defined by:

(Def.22) $\langle {}^{\textcircled{a}}(X, t), \text{the carrier of } S \rangle = t$.

Let S be a non void non empty many sorted signature, let U_0 be a non-empty algebra over S , let o be an operation symbol of S , and let p be a finite sequence. Let us assume that $p \in \text{Args}(o, U_0)$. The functor $\pi(o, U_0, p)$ yielding an element of $\bigcup (\text{the sorts of } U_0)$ is defined by:

(Def.23) $\pi(o, U_0, p) = (\text{Den}(o, U_0))(p)$.

Next we state two propositions:

(17) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S . Then $\text{FreeGenerator}(X)$ is free.

(18) Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set of the carrier of S . Then $\text{Free}(X)$ is free.

Let S be a non void non empty many sorted signature. One can check that there exists a non-empty algebra over S which is free and strict.

Let S be a non void non empty many sorted signature and let U_0 be a free non-empty algebra over S . One can verify that there exists a generator set of U_0 which is free.

One can prove the following propositions:

- (19) Let S be a non void non empty many sorted signature and let U_1 be a non-empty algebra over S . Then there exists a strict free non-empty algebra U_0 over S such that there exists many sorted function from U_0 into U_1 which is an epimorphism of U_0 onto U_1 .
- (20) Let S be a non void non empty many sorted signature and let U_1 be a strict non-empty algebra over S . Then there exists a strict free non-empty algebra U_0 over S and there exists a many sorted function F from U_0 into U_1 such that F is an epimorphism of U_0 onto U_1 and $\text{Im } F = U_1$.

REFERENCES

- [1] Grzegorz Bancerek. Cartesian product of functions. *Formalized Mathematics*, 2(4):547–552, 1991.
- [2] Grzegorz Bancerek. Introduction to trees. *Formalized Mathematics*, 1(2):421–427, 1990.
- [3] Grzegorz Bancerek. Joining of decorated trees. *Formalized Mathematics*, 4(1):77–82, 1993.
- [4] Grzegorz Bancerek. König’s lemma. *Formalized Mathematics*, 2(3):397–402, 1991.
- [5] Grzegorz Bancerek. König’s theorem. *Formalized Mathematics*, 1(3):589–593, 1990.
- [6] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Formalized Mathematics*, 3(2):195–204, 1992.
- [7] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [8] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Formalized Mathematics*, 4(1):91–101, 1993.
- [9] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Formalized Mathematics*, 5(1):47–54, 1996.
- [10] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(3):529–536, 1990.
- [11] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [12] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [13] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [14] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar - part 1. *Formalized Mathematics*, 2(5):683–687, 1991.
- [15] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Formalized Mathematics*, 5(1):61–65, 1996.
- [16] Jarosław Kotowicz, Beata Madras, and Małgorzata Korolkiewicz. Basic notation of universal algebra. *Formalized Mathematics*, 3(2):251–253, 1992.
- [17] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(1):103–108, 1993.
- [18] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [19] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(1):37–42, 1996.
- [20] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(1):15–22, 1993.
- [21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(1):9–11, 1990.
- [22] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(1):97–105, 1990.
- [23] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(3):575–579, 1990.

- [24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [25] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

Received April 27, 1994
