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Many Sorted Algebras

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Summary. The basic purpose of the paper is to prepare preliminaries of the theory of many sorted algebras. The concept of the signature of a many sorted algebra is introduced as well as the concept of many sorted algebra itself. Some auxiliary related notions are defined. The correspondence between (1 sorted) universal algebras [9] and many sorted algebras with one sort only is described by introducing two functors mapping one into the other. The construction is done this way that the composition of both functors is the identity on universal algebras.

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The articles [12], [14], [5], [6], [2], [10], [7], [4], [1], [11], [13], [3], [8], and [9] provide the notation and terminology for this paper.

1. Preliminaries

In this paper i, j are arbitrary and I is a set.

Next we state the proposition

(1) It is not true that there exists a non-empty many sorted set M of I such that $\emptyset \in \operatorname{rng} M$.

In this article we present several logical schemes. The scheme *MSSEx* deals with a set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a many sorted set f of $\mathcal{A}$ such that for every i such that $i \in \mathcal{A}$ holds $\mathcal{P}[i, f(i)]$

provided the following condition is met:

- For every i such that $i \in \mathcal{A}$ there exists j such that $\mathcal{P}[i, j]$.

The scheme *MSSLambda* concerns a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

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There exists a many sorted set f of $\mathcal{A}$ such that for every i such that $i \in \mathcal{A}$ holds $f(i) = \mathcal{F}(i)$

for all values of the parameters.

Let I be a set and let M be a many sorted set of I. A component of M is an element of $\operatorname{rng} M$.

Next we state two propositions:

(2) Let I be a non empty set, and let M be a many sorted set of I, and let A be a component of M. Then there exists i such that $i \in I$ and $A = M(i)$.

(3) For every many sorted set M of I and for every i such that $i \in I$ holds $M(i)$ is a component of M.

Let us consider I and let B be a many sorted set of I. A many sorted set of I is said to be an element of B if:

(Def.1) For every i such that $i \in I$ holds $\mathrm{it}(i)$ is an element of $B(i)$.

2. Auxiliary functors

Let us consider I, let A be a many sorted set of I, and let B be a many sorted set of I. A many sorted set of I is called a many sorted function from A into B if:

(Def.2) For every i such that $i \in I$ holds $\mathrm{it}(i)$ is a function from $A(i)$ into $B(i)$.

Let us consider I, let A be a many sorted set of I, and let B be a many sorted set of I. Note that every many sorted function from A into B is function yielding.

Let I be a set and let M be a many sorted set of I. The functor $M^{\#}$ yielding a many sorted set of I^* is defined by:

(Def.3) For every element i of I^* holds $M^{\#}(i) = \prod(M \cdot i)$.

Let I be a set and let M be a non-empty many sorted set of I. Note that $M^{\#}$ is non-empty.

Let us consider I, let J be a non empty set, let O be a function from I into J, and let F be a many sorted set of J. Then $F \cdot O$ is a many sorted set of I.

Let us consider I, let J be a non empty set, let O be a function from I into J, and let F be a non-empty many sorted set of J. Then $F \cdot O$ is a non-empty many sorted set of I.

Let a be arbitrary. The functor $\square \longmapsto a$ yields a function from $\mathbb{N}$ into $\{a\}^*$ and is defined as follows:

(Def.4) For every natural number n holds $(\square \longmapsto a)(n) = n \mapsto a$.

In the sequel D denotes a non empty set and n denotes a natural number.

The following propositions are true:

(4) For arbitrary a, b holds $(\{a\} \longmapsto b) \cdot (n \mapsto a) = n \mapsto b$.

(5) For arbitrary a and for every many sorted set M of $\{a\}$ such that $M = \{a\} \longmapsto D$ holds $(M^{\#} \cdot (\square \longmapsto a))(n) = D^{\operatorname{Seg} n}$.

Let us consider I, i. Then $I \longmapsto i$ is a function from I into $\{i\}$.

Let C be a set, let A, B be non empty sets, let F be a partial function from C to A, and let G be a function from A into B. Then $G \cdot F$ is a function from $\mathrm{dom}\, F$ into B.

3. Many sorted signatures

We introduce many sorted signatures which are extensions of 1-sorted structure and are systems

$\langle$ a carrier, operation symbols, an arity, a result sort $\rangle$,

where the carrier is a set, the operation symbols constitute a set, the arity is a function from the operation symbols into the carrier*, and the result sort is a function from the operation symbols into the carrier.

A many sorted signature is void if:

(Def.5) The operation symbols of it $= \emptyset$.

One can verify that there exists a many sorted signature which is void strict and non empty and there exists a many sorted signature which is non void strict and non empty.

In the sequel S is a non empty many sorted signature.

Let us consider S. A sort symbol of S is an element of the carrier of S. An operation symbol of S is an element of the operation symbols of S.

Let S be a non void non empty many sorted signature and let o be an operation symbol of S. The functor $\mathrm{Arity}(o)$ yields an element of (the carrier of S)* and is defined as follows:

(Def.6) $\mathrm{Arity}(o) =$ (the arity of S)(o).

The result sort of o yielding an element of the carrier of S is defined by:

(Def.7) The result sort of $o =$ (the result sort of S)(o).

4. Many sorted algebras

Let S be a 1-sorted structure. We consider many-sorted structures over S as systems

$\langle$ sorts $\rangle$,

where the sorts constitute a many sorted set of the carrier of S.

Let us consider S. We consider algebras over S as extensions of many-sorted structure over S as systems

$\langle$ sorts, a characteristics $\rangle$,

where the sorts constitute a many sorted set of the carrier of S and the characteristics is a many sorted function from the sorts$^\#$ $\cdot$ (the arity of S) into (the sorts) $\cdot$(the result sort of S).

Let us consider S and let A be an algebra over S. We say that A is non-empty if and only if:

(Def.8) The sorts of A is non-empty.

Let us consider S. Observe that there exists an algebra over S which is strict and non-empty.

Let us consider S and let A be a non-empty algebra over S. One can verify that the sorts of A is non-empty.

Let us consider S and let A be a non-empty algebra over S. One can check that every component of the sorts of A is non empty and every component of the sorts of $A^{\#}$ is non empty.

Let S be a non void non empty many sorted signature, let o be an operation symbol of S, and let A be an algebra over S. The functor $\mathrm{Args}(o, A)$ yielding a component of (the sorts of $A)^{\#}$ is defined by:

(Def.9) $\mathrm{Args}(o, A) = ((\text{the sorts of } A)^{\#} \cdot (\text{the arity of } S))(o)$.

The functor $\mathrm{Result}(o, A)$ yields a component of the sorts of A and is defined as follows:

(Def.10) $\mathrm{Result}(o, A) = ((\text{the sorts of } A) \cdot (\text{the result sort of } S))(o)$.

Let S be a non void non empty many sorted signature, let o be an operation symbol of S, and let A be an algebra over S. The functor $\mathrm{Den}(o, A)$ yielding a function from $\mathrm{Args}(o, A)$ into $\mathrm{Result}(o, A)$ is defined as follows:

(Def.11) $\mathrm{Den}(o, A) = (\text{the characteristics of } A)(o)$.

The following proposition is true

(6) Let S be a non void non empty many sorted signature, and let o be an operation symbol of S, and let A be a non-empty algebra over S. Then $\mathrm{Den}(o, A)$ is non empty.

5. Universal algebras as many sorted

We now state two propositions:

(8)[1] For every homogeneous quasi total non empty partial function h from D^* to D holds $\mathrm{dom}\, h = D^{\mathrm{Seg}\,\mathrm{arity}\, h}$.

(9) For every universal algebra A holds signature A is non empty.

6. Universal algebras for many sorted algebras with one sort

Let A be a universal algebra. Then signature A is a finite sequence of elements of $\mathbb{N}$.

A many sorted signature is segmental if:

(Def.12) There exists n such that the operation symbols of it $= \mathrm{Seg}\, n$.

The following proposition is true

[1]The proposition (7) has been removed.

(10) Let S be a non empty many sorted signature. Suppose S is trivial. Let A be an algebra over S and let c_1, c_2 be components of the sorts of A. Then $c_1 = c_2$.

Let us mention that there exists a many sorted signature which is segmental trivial non void strict and non empty.

Let A be a universal algebra. The functor MSSign(A) yields a non void strict segmental trivial many sorted signature and is defined by:

(Def.13) $\mathrm{MSSign}(A) = \langle\{0\}, \mathrm{dom\,signature}\, A, (\square \longmapsto 0) \cdot \mathrm{signature}\, A, \mathrm{dom\,signature}\, A \longmapsto 0\rangle$.

Let A be a universal algebra. One can check that MSSign(A) is non empty.

Let A be a universal algebra. The functor MSSorts(A) yields a non-empty many sorted set of the carrier of MSSign(A) and is defined as follows:

(Def.14) $\mathrm{MSSorts}(A) = \{0\} \longmapsto$ the carrier of A.

Let A be a universal algebra. The functor MSCharact(A) yields a many sorted function from $(\mathrm{MSSorts}(A))^{\#} \cdot$(the arity of MSSign($A$)) into $\mathrm{MSSorts}(A) \cdot$ (the result sort of MSSign(A)) and is defined by:

(Def.15) MSCharact(A) = the characteristic of A.

Let A be a universal algebra. The functor MSAlg(A) yielding a strict algebra over MSSign(A) is defined by:

(Def.16) $\mathrm{MSAlg}(A) = \langle\mathrm{MSSorts}(A), \mathrm{MSCharact}(A)\rangle$.

Let A be a universal algebra. Note that MSAlg(A) is non-empty.

Let M_1 be a trivial non empty many sorted signature and let A be an algebra over M_1. The sort of A yielding a set is defined as follows:

(Def.17) There exists a component c of the sorts of A such that the sort of $A = c$.

Let M_1 be a trivial non empty many sorted signature and let A be a non-empty algebra over M_1. Observe that the sort of A is non empty.

We now state four propositions:

(11) Let M_1 be a segmental trivial non void non empty many sorted signature, and let i be an operation symbol of M_1, and let A be a non-empty algebra over M_1. Then $\mathrm{Args}(i, A) = (\text{the sort of } A)^{\mathrm{len\,Arity}(i)}$.

(12) For every non empty set A and for every n holds $A^n \subseteq A^*$.

(13) Let M_1 be a segmental trivial non void non empty many sorted signature, and let i be an operation symbol of M_1, and let A be a non-empty algebra over M_1. Then $\mathrm{Args}(i, A) \subseteq (\text{the sort of } A)^*$.

(14) Let M_1 be a segmental trivial non void non empty many sorted signature and let A be a non-empty algebra over M_1. Then the characteristics of A is a finite sequence of elements of $(\text{the sort of } A)^* \dot{\rightarrow} \text{the sort of } A$.

Let M_1 be a segmental trivial non void non empty many sorted signature and let A be a non-empty algebra over M_1. The functor charact(A) yielding a finite sequence of operational functions of the sort of A is defined by:

(Def.18) charact(A) = the characteristics of A.

In the sequel M_1 will denote a segmental trivial non void non empty many sorted signature and A will denote a non-empty algebra over M_1.

Let us consider M_1, A. The functor $\mathrm{Alg}_{\mathbf{1}}(A)$ yields a non-empty strict universal algebra and is defined as follows:

(Def.19) $\mathrm{Alg}_{\mathbf{1}}(A) = \langle$the sort of A, $\mathrm{charact}(A)\rangle$.

We now state the proposition

(15) For every strict universal algebra A holds $A = \mathrm{Alg}_{\mathbf{1}}(\mathrm{MSAlg}(A))$.

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