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Subalgebras of Many Sorted Algebra. Lattice of Subalgebras

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The articles [12], [13], [5], [6], [2], [8], [9], [7], [4], [14], [3], [1], [11], and [10] provide the notation and terminology for this paper.

1. Auxilary Facts about Many Sorted Sets

In this paper x will be arbitrary.

The scheme *LambdaB* concerns a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

There exists a function f such that $\mathrm{dom}\, f = \mathcal{A}$ and for every element d of $\mathcal{A}$ holds $f(d) = \mathcal{F}(d)$

for all values of the parameters.

Let I be a set, let X be a many sorted set of I, and let Y be a non-empty many sorted set of I. Observe that $X \cup Y$ is non-empty and $Y \cup X$ is non-empty.

Next we state two propositions:

(1) Let I be a set, and let X be a many sorted set of I, and let Y be a non-empty many sorted set of I. Then $X \cup Y$ is non-empty and $Y \cup X$ is non-empty.

(2) For every non empty set I and for all many sorted sets X, Y of I and for every element i of I^* holds $\prod((X \cap Y) \cdot i) = \prod(X \cdot i) \cap \prod(Y \cdot i)$.

Let I be a set and let M be a many sorted set of I. A many sorted set of I is said to be a many sorted subset of M if:

(Def.1) It $\subseteq M$.

Let I be a set and let M be a non-empty many sorted set of I. Observe that there exists a many sorted subset of M which is non-empty.

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2. Constants of a Many Sorted Algebra

We follow the rules: S will denote a non void non empty many sorted signature, o will denote an operation symbol of S, and U_0, U_1, U_2 will denote algebras over S.

Let S be a non empty many sorted signature and let U_0 be an algebra over S. A subset of U_0 is a many sorted subset of the sorts of U_0.

Let S be a non empty many sorted signature. A sort symbol of S has constants if:

(Def.2) There exists an operation symbol o of S such that (the arity of S)(o) $= \varepsilon$ and (the result sort of S)(o) $=$ it.

A non empty many sorted signature has constant operations if:

(Def.3) Every sort symbol of it has constants.

Let A be a non empty set, let B be a set, let a be a function from B into A^*, and let r be a function from B into A. Note that $\langle A, B, a, r\rangle$ is non empty.

Let us observe that there exists a non empty many sorted signature which is non void and strict and has constant operations.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let s be a sort symbol of S. The functor Constants(U_0, s) yielding a subset of (the sorts of U_0)(s) is defined by:

(Def.4) (i) There exists a non empty set A such that $A =$ (the sorts of U_0)(s) and Constants(U_0, s) $= \{a : a$ ranges over elements of A, $\bigvee_o$ (the arity of S)(o) $= \varepsilon \ \wedge$ (the result sort of S)(o) $= s \ \wedge \ a \in \operatorname{rng} \operatorname{Den}(o, U_0)\}$ if (the sorts of U_0)(s) $\neq \emptyset$,

(ii) Constants(U_0, s) $= \emptyset$, otherwise.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. The functor Constants(U_0) yielding a subset of U_0 is defined as follows:

(Def.5) For every sort symbol s of S holds (Constants(U_0))(s) $=$ Constants(U_0, s).

Let S be a non void non empty many sorted signature with constant operations, let U_0 be a non-empty algebra over S, and let s be a sort symbol of S. One can verify that Constants(U_0, s) is non empty.

Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. One can verify that Constants(U_0) is non-empty.

3. Subalgebras of a Many Sorted Algebra

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, let o be an operation symbol of S, and let A be a subset of U_0. We say that A is closed on o if and only if:

(Def.6) $\mathrm{rng}(\mathrm{Den}(o, U_0) \upharpoonright (A^{\#} \cdot (\text{the arity of } S))(o)) \subseteq (A \cdot (\text{the result sort of } S))(o)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. We say that A is operations closed if and only if:

(Def.7) For every operation symbol o of S holds A is closed on o.

One can prove the following proposition

(3) Let S be a non void non empty many sorted signature, and let o be an operation symbol of S, and let U_0 be an algebra over S, and let B_0, B_1 be subsets of U_0. If $B_0 \subseteq B_1$, then $({B_0}^{\#} \cdot (\text{the arity of } S))(o) \subseteq ({B_1}^{\#} \cdot (\text{the arity of } S))(o)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, let o be an operation symbol of S, and let A be a subset of U_0. Let us assume that A is closed on o. The functor o_A yielding a function from $(A^{\#} \cdot (\text{the arity of } S))(o)$ into $(A \cdot (\text{the result sort of } S))(o)$ is defined as follows:

(Def.8) $o_A = \mathrm{Den}(o, U_0) \upharpoonright (A^{\#} \cdot (\text{the arity of } S))(o)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. The functor $\mathrm{Opers}(U_0, A)$ yielding a many sorted function from $A^{\#} \cdot (\text{the arity of } S)$ into $A \cdot (\text{the result sort of } S)$ is defined by:

(Def.9) For every operation symbol o of S holds $(\mathrm{Opers}(U_0, A))(o) = o_A$.

Next we state two propositions:

(4) Let U_0 be an algebra over S and let B be a subset of U_0. Suppose $B =$ the sorts of U_0. Then B is operations closed and for every o holds $o_B = \mathrm{Den}(o, U_0)$.

(5) For every subset B of U_0 such that $B =$ the sorts of U_0 holds $\mathrm{Opers}(U_0, B) =$ the characteristics of U_0.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. An algebra over S is called a subalgebra of U_0 if it satisfies the conditions (Def.10).

(Def.10) (i) The sorts of it is a subset of U_0, and

(ii) for every subset B of U_0 such that $B =$ the sorts of it holds B is operations closed and the characteristics of it $= \mathrm{Opers}(U_0, B)$.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. One can check that there exists a subalgebra of U_0 which is strict.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. Observe that there exists a subalgebra of U_0 which is non-empty and strict.

One can prove the following propositions:

(6) U_0 is a subalgebra of U_0.

(7) If U_0 is a subalgebra of U_1 and U_1 is a subalgebra of U_2, then U_0 is a subalgebra of U_2.

(8) If U_1 is a strict subalgebra of U_2 and U_2 is a strict subalgebra of U_1, then $U_1 = U_2$.

(9) For all subalgebras U_1, U_2 of U_0 such that the sorts of $U_1 \subseteq$ the sorts of U_2 holds U_1 is a subalgebra of U_2.

(10) For all strict subalgebras U_1, U_2 of U_0 such that the sorts of $U_1 =$ the sorts of U_2 holds $U_1 = U_2$.

(11) Let S be a non void non empty many sorted signature, and let U_0 be an algebra over S, and let U_1 be a subalgebra of U_0. Then Constants(U_0) is a subset of U_1.

(12) Let S be a non void non empty many sorted signature with constant operations, and let U_0 be a non-empty algebra over S, and let U_1 be a non-empty subalgebra of U_0. Then Constants(U_0) is a non-empty subset of U_1.

(13) Let S be a non void non empty many sorted signature with constant operations, and let U_0 be a non-empty algebra over S, and let U_1, U_2 be non-empty subalgebras of U_0. Then (the sorts of U_1) $\cap$ (the sorts of U_2) is non-empty.

4. Many Sorted Subsets of Many Sorted Algebra

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. The functor SubSorts(A) yielding a non empty set is defined by the condition (Def.11).

(Def.11) Let x be arbitrary. Then $x \in$ SubSorts(A) if and only if the following conditions are satisfied:

(i) $x \in (2^{\bigcup (\text{the sorts of } U_0)})^{\text{the carrier of } S}$,

(ii) x is a subset of U_0, and

(iii) for every subset B of U_0 such that $B = x$ holds B is operations closed and Constants(U_0) $\subseteq B$ and $A \subseteq B$.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. The functor SubSorts(U_0) yields a non empty set and is defined by the condition (Def.12).

(Def.12) Let x be arbitrary. Then $x \in$ SubSorts(U_0) if and only if the following conditions are satisfied:

(i) $x \in (2^{\bigcup (\text{the sorts of } U_0)})^{\text{the carrier of } S}$,

(ii) x is a subset of U_0, and

(iii) for every subset B of U_0 such that $B = x$ holds B is operations closed.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let e be an element of SubSorts(U_0). The functor $^{@}e$ yielding a subset of U_0 is defined as follows:

(Def.13) $^{@}e = e$.

Next we state two propositions:

(14) For all subsets A, B of U_0 holds $B \in \mathrm{SubSorts}(A)$ iff B is operations closed and $\mathrm{Constants}(U_0) \subseteq B$ and $A \subseteq B$.

(15) For every subset B of U_0 holds $B \in \mathrm{SubSorts}(U_0)$ iff B is operations closed.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, let A be a subset of U_0, and let s be a sort symbol of S. The functor $\mathrm{SubSort}(A, s)$ yields a non empty set and is defined as follows:

(Def.14) For arbitrary x holds $x \in \mathrm{SubSort}(A, s)$ iff there exists a subset B of U_0 such that $B \in \mathrm{SubSorts}(A)$ and $x = B(s)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. The functor $\mathrm{MSSubSort}(A)$ yields a subset of U_0 and is defined as follows:

(Def.15) For every sort symbol s of S holds $(\mathrm{MSSubSort}(A))(s) = \bigcap \mathrm{SubSort}(A, s)$.

We now state several propositions:

(16) For every subset A of U_0 holds $\mathrm{Constants}(U_0) \cup A \subseteq \mathrm{MSSubSort}(A)$.

(17) For every subset A of U_0 such that $\mathrm{Constants}(U_0) \cup A$ is non-empty holds $\mathrm{MSSubSort}(A)$ is non-empty.

(18) Let A be a subset of U_0 and let B be a subset of U_0. If $B \in \mathrm{SubSorts}(A)$, then $((\mathrm{MSSubSort}(A))^{\#} \cdot (\text{the arity of } S))(o) \subseteq (B^{\#} \cdot (\text{the arity of } S))(o)$.

(19) Let A be a subset of U_0 and let B be a subset of U_0. Suppose $B \in \mathrm{SubSorts}(A)$. Then $\mathrm{rng}(\mathrm{Den}(o, U_0) \restriction ((\mathrm{MSSubSort}(A))^{\#} \cdot (\text{the arity of } S))(o)) \subseteq (B \cdot (\text{the result sort of } S))(o)$.

(20) For every subset A of U_0 holds $\mathrm{rng}(\mathrm{Den}(o, U_0) \restriction ((\mathrm{MSSubSort}(A))^{\#} \cdot (\text{the arity of } S))(o)) \subseteq (\mathrm{MSSubSort}(A) \cdot (\text{the result sort of } S))(o)$.

(21) For every subset A of U_0 holds $\mathrm{MSSubSort}(A)$ is operations closed and $A \subseteq \mathrm{MSSubSort}(A)$.

5. Operations on Many Sorted Algebra and its Subalgebras

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. Let us assume that A is operations closed. The functor $U_0 \restriction A$ yields a strict subalgebra of U_0 and is defined as follows:

(Def.16) $U_0 \restriction A = \langle A, (\mathrm{Opers}(U_0, A)$ **qua** many sorted function from $A^{\#} \cdot (\text{the arity of } S)$ into $A \cdot (\text{the result sort of } S))\rangle$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let U_1, U_2 be subalgebras of U_0. The functor $U_1 \cap U_2$ yielding a strict subalgebra of U_0 is defined by the conditions (Def.17).

(Def.17) (i) The sorts of $U_1 \cap U_2 =$ (the sorts of U_1) $\cap$ (the sorts of U_2), and

(ii) for every subset B of U_0 such that $B =$ the sorts of $U_1 \cap U_2$ holds B is operations closed and the characteristics of $U_1 \cap U_2 = \mathrm{Opers}(U_0, B)$.

Let S be a non void non empty many sorted signature, let U_0 be an algebra over S, and let A be a subset of U_0. The functor $\mathrm{Gen}(A)$ yields a strict subalgebra of U_0 and is defined by the conditions (Def.18).

(Def.18) (i) A is a subset of $\mathrm{Gen}(A)$, and

(ii) for every subalgebra U_1 of U_0 such that A is a subset of U_1 holds $\mathrm{Gen}(A)$ is a subalgebra of U_1.

Let S be a non void non empty many sorted signature, let U_0 be a non-empty algebra over S, and let A be a non-empty subset of U_0. Observe that $\mathrm{Gen}(A)$ is non-empty.

We now state three propositions:

(22) Let S be a non void non empty many sorted signature, and let U_0 be a strict algebra over S, and let B be a subset of U_0. If $B =$ the sorts of U_0, then $\mathrm{Gen}(B) = U_0$.

(23) Let S be a non void non empty many sorted signature, and let U_0 be an algebra over S, and let U_1 be a strict subalgebra of U_0, and let B be a subset of U_0. If $B =$ the sorts of U_1, then $\mathrm{Gen}(B) = U_1$.

(24) Let S be a non void non empty many sorted signature with constant operations, and let U_0 be a non-empty algebra over S, and let U_1 be a subalgebra of U_0. Then $\mathrm{Gen}(\mathrm{Constants}(U_0)) \cap U_1 = \mathrm{Gen}(\mathrm{Constants}(U_0))$.

Let S be a non void non empty many sorted signature, let U_0 be a non-empty algebra over S, and let U_1, U_2 be subalgebras of U_0. The functor $U_1 \bigsqcup U_2$ yielding a strict subalgebra of U_0 is defined as follows:

(Def.19) For every subset A of U_0 such that $A =$ (the sorts of U_1) $\cup$ (the sorts of U_2) holds $U_1 \bigsqcup U_2 = \mathrm{Gen}(A)$.

Next we state several propositions:

(25) Let S be a non void non empty many sorted signature, and let U_0 be a non-empty algebra over S, and let U_1 be a subalgebra of U_0, and let A, B be subsets of U_0. If $B = A \cup$ the sorts of U_1, then $\mathrm{Gen}(A) \bigsqcup U_1 = \mathrm{Gen}(B)$.

(26) Let S be a non void non empty many sorted signature, and let U_0 be a non-empty algebra over S, and let U_1 be a subalgebra of U_0, and let B be a subset of U_0. If $B =$ the sorts of U_0, then $\mathrm{Gen}(B) \bigsqcup U_1 = \mathrm{Gen}(B)$.

(27) Let S be a non void non empty many sorted signature, and let U_0 be a non-empty algebra over S, and let U_1, U_2 be subalgebras of U_0. Then $U_1 \bigsqcup U_2 = U_2 \bigsqcup U_1$.

(28) Let S be a non void non empty many sorted signature, and let U_0 be a non-empty algebra over S, and let U_1, U_2 be strict subalgebras of U_0. Then $U_1 \cap (U_1 \bigsqcup U_2) = U_1$.

(29) Let S be a non void non empty many sorted signature with constant operations, and let U_0 be a non-empty algebra over S, and let U_1, U_2 be strict subalgebras of U_0. Then $U_1 \cap U_2 \bigsqcup U_2 = U_2$.

6. Lattice of Subalgebras of Many Sorted Algebra

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. The functor Subalgebras(U_0) yielding a non empty set is defined as follows:

(Def.20) For every x holds $x \in$ Subalgebras(U_0) iff x is a strict subalgebra of U_0.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. The functor MSAlgJoin(U_0) yields a binary operation on Subalgebras(U_0) and is defined by:

(Def.21) For all elements x, y of Subalgebras(U_0) and for all strict subalgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds (MSAlgJoin(U_0))(x, y) $= U_1 \bigsqcup U_2$.

Let S be a non void non empty many sorted signature and let U_0 be a non-empty algebra over S. The functor MSAlgMeet(U_0) yielding a binary operation on Subalgebras(U_0) is defined by:

(Def.22) For all elements x, y of Subalgebras(U_0) and for all strict subalgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds (MSAlgMeet(U_0))(x, y) $= U_1 \cap U_2$.

In the sequel U_0 is a non-empty algebra over S.

We now state four propositions:

(30) MSAlgJoin(U_0) is commutative.

(31) MSAlgJoin(U_0) is associative.

(32) Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. Then MSAlgMeet(U_0) is commutative.

(33) Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. Then MSAlgMeet(U_0) is associative.

Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. The lattice of subalgebras of U_0 yields a strict lattice and is defined as follows:

(Def.23) The lattice of subalgebras of $U_0 = \langle$Subalgebras(U_0), MSAlgJoin(U_0), MSAlgMeet$(U_0)\rangle$.

The following proposition is true

(34) Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. Then the lattice of subalgebras of U_0 is bounded.

Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. Note that the lattice of subalgebras of U_0 is bounded.

We now state three propositions:

(35) Let S be a non void non empty many sorted signature with constant operations and let U_0 be a non-empty algebra over S. Then $\bot_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}(\mathrm{Constants}(U_0))$.

(36) Let S be a non void non empty many sorted signature with constant operations, and let U_0 be a non-empty algebra over S, and let B be a subset of U_0. If $B =$ the sorts of U_0, then $\top_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}(B)$.

(37) Let S be a non void non empty many sorted signature with constant operations and let U_0 be a strict non-empty algebra over S. Then $\top_{\text{the lattice of subalgebras of } U_0} = U_0$.

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