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Homomorphisms of Many Sorted Algebras

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Summary. The aim of this article is to present the definition and some properties of homomorphisms of many sorted algebras. Some auxiliary properties of many sorted functions also have been shown.

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The notation and terminology used in this paper have been introduced in the following articles: [10], [12], [13], [5], [6], [2], [4], [1], [11], [9], [7], [8], and [3].

1. Preliminaries

For simplicity we follow the rules: S is a non void non empty many sorted signature, U_1, U_2, U_3 are non-empty algebras over S, o is an operation symbol of S, and n is a natural number.

Let I be a non empty set, let A, B be non-empty many sorted sets of I, let F be a many sorted function from A into B, and let i be an element of I. Then $F(i)$ is a function from $A(i)$ into $B(i)$.

Let us consider S, U_1, U_2. A many sorted function from U_1 into U_2 is a many sorted function from the sorts of U_1 into the sorts of U_2.

Let I be a set and let A be a many sorted set of I. The functor id_A yields a many sorted function from A into A and is defined as follows:

(Def.1) For arbitrary i such that $i \in I$ holds $\mathrm{id}_A(i) = \mathrm{id}_{A(i)}$.

A function is "1-1" if:

(Def.2) For arbitrary i and for every function f such that $i \in \mathrm{dom}\,\mathrm{it}$ and $\mathrm{it}(i) = f$ holds f is one-to-one.

Let I be a set. Observe that there exists a many sorted function of I which is "1-1".

We now state the proposition

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(1) Let I be a set and let F be a many sorted function of I. Then F is "1-1" if and only if for arbitrary i and for every function f such that $i \in I$ and $F(i) = f$ holds f is one-to-one.

Let I be a set and let A, B be many sorted sets of I. A many sorted function from A into B is "onto" if:

(Def.3) For arbitrary i and for every function f from $A(i)$ into $B(i)$ such that $i \in I$ and $\mathrm{it}(i) = f$ holds $\mathrm{rng}\, f = B(i)$.

Let F, G be function yielding functions. The functor $G \circ F$ yielding a function yielding function is defined by the conditions (Def.4).

(Def.4) (i) $\mathrm{dom}(G \circ F) = \mathrm{dom}\, F \cap \mathrm{dom}\, G$, and

(ii) for arbitrary i and for every function f and for every function g such that $i \in \mathrm{dom}(G \circ F)$ and $f = F(i)$ and $g = G(i)$ holds $(G \circ F)(i) = g \cdot f$.

We now state the proposition

(2) Let I be a set, and let A be a many sorted set of I, and let B, C be non-empty many sorted sets of I, and let F be a many sorted function from A into B, and let G be a many sorted function from B into C. Then

(i) $\mathrm{dom}(G \circ F) = I$, and

(ii) for arbitrary i and for every function f from $A(i)$ into $B(i)$ and for every function g from $B(i)$ into $C(i)$ such that $i \in I$ and $f = F(i)$ and $g = G(i)$ holds $(G \circ F)(i) = g \cdot f$.

Let I be a set, let A be a many sorted set of I, let B, C be non-empty many sorted sets of I, let F be a many sorted function from A into B, and let G be a many sorted function from B into C. Then $G \circ F$ is a many sorted function from A into C.

Next we state two propositions:

(3) Let I be a set, and let A, B be non-empty many sorted sets of I, and let F be a many sorted function from A into B. Then $F \circ \mathrm{id}_A = F$.

(4) Let I be a set, and let A be a many sorted set of I, and let B be a non-empty many sorted set of I, and let F be a many sorted function from A into B. Then $\mathrm{id}_B \circ F = F$.

Let I be a set, let A, B be non-empty many sorted sets of I, and let F be a many sorted function from A into B. Let us assume that F is "1-1" and "onto". The functor F^{-1} yielding a many sorted function from B into A is defined as follows:

(Def.5) For arbitrary i and for every function f from $A(i)$ into $B(i)$ such that $i \in I$ and $f = F(i)$ holds $F^{-1}(i) = f^{-1}$.

We now state the proposition

(5) Let I be a set, and let A, B be non-empty many sorted sets of I, and let H be a many sorted function from A into B, and let H_1 be a many sorted function from B into A. If H is "1-1" and "onto" and $H_1 = H^{-1}$, then $H \circ H_1 = \mathrm{id}_B$ and $H_1 \circ H = \mathrm{id}_A$.

Let I be a set, let A be a many sorted set of I, and let F be a many sorted function of I. The functor $F \circ A$ yields a many sorted set of I and is defined as follows:

(Def.6) For arbitrary i and for every function f such that $i \in I$ and $f = F(i)$ holds $(F \circ A)(i) = f^{\circ}A(i)$.

Let us consider S, U_1, o. Observe that every element of $\mathrm{Args}(o, U_1)$ is function-like and relation-like.

2. Homomorphisms of Many Sorted Algebras

One can prove the following proposition

(6) Let x be an element of $\mathrm{Args}(o, U_1)$. Then $\mathrm{dom}\, x = \mathrm{dom}\, \mathrm{Arity}(o)$ and for arbitrary y such that $y \in \mathrm{dom}((\text{the sorts of } U_1) \cdot \mathrm{Arity}(o))$ holds $x(y) \in ((\text{the sorts of } U_1) \cdot \mathrm{Arity}(o))(y)$.

Let us consider S, U_1, U_2, o, let F be a many sorted function from U_1 into U_2, and let x be an element of $\mathrm{Args}(o, U_1)$. The functor $F\#x$ yielding an element of $\mathrm{Args}(o, U_2)$ is defined by:

(Def.7) For every n such that $n \in \mathrm{dom}\, x$ holds $(F\#x)(n) = F(\pi_n \mathrm{Arity}(o))(x(n))$.

The following two propositions are true:

(7) For all S, o, U_1 and for every element x of $\mathrm{Args}(o, U_1)$ holds $x = \mathrm{id}_{(\text{the sorts of } U_1)}\#x$.

(8) Let H_1 be a many sorted function from U_1 into U_2, and let H_2 be a many sorted function from U_2 into U_3, and let x be an element of $\mathrm{Args}(o, U_1)$. Then $(H_2 \circ H_1)\#x = H_2\#(H_1\#x)$.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. We say that F is a homomorphism of U_1 into U_2 if and only if:

(Def.8) For every operation symbol o of S and for every element x of $\mathrm{Args}(o, U_1)$ holds $F(\text{the result sort of } o)((\mathrm{Den}(o, U_1))(x)) = (\mathrm{Den}(o, U_2))(F\#x)$.

Next we state two propositions:

(9) $\mathrm{id}_{(\text{the sorts of } U_1)}$ is a homomorphism of U_1 into U_1.

(10) Let H_1 be a many sorted function from U_1 into U_2 and let H_2 be a many sorted function from U_2 into U_3. Suppose H_1 is a homomorphism of U_1 into U_2 and H_2 is a homomorphism of U_2 into U_3. Then $H_2 \circ H_1$ is a homomorphism of U_1 into U_3.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. We say that F is an epimorphism of U_1 onto U_2 if and only if:

(Def.9) F is a homomorphism of U_1 into U_2 and "onto".

One can prove the following proposition

(11) Let F be a many sorted function from U_1 into U_2 and let G be a many sorted function from U_2 into U_3. Suppose F is an epimorphism of U_1 onto U_2 and G is an epimorphism of U_2 onto U_3. Then $G \circ F$ is an epimorphism of U_1 onto U_3.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. We say that F is a monomorphism of U_1 into U_2 if and only if:

(Def.10) F is a homomorphism of U_1 into U_2 and "1-1".

The following proposition is true

(12) Let F be a many sorted function from U_1 into U_2 and let G be a many sorted function from U_2 into U_3. Suppose F is a monomorphism of U_1 into U_2 and G is a monomorphism of U_2 into U_3. Then $G \circ F$ is a monomorphism of U_1 into U_3.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. We say that F is an isomorphism of U_1 and U_2 if and only if:

(Def.11) F is an epimorphism of U_1 onto U_2 and a monomorphism of U_1 into U_2.

The following propositions are true:

(13) Let F be a many sorted function from U_1 into U_2. Then F is an isomorphism of U_1 and U_2 if and only if F is a homomorphism of U_1 into U_2 "onto" and "1-1".

(14) Let H be a many sorted function from U_1 into U_2 and let H_1 be a many sorted function from U_2 into U_1. Suppose H is an isomorphism of U_1 and U_2 and $H_1 = H^{-1}$. Then H_1 is an isomorphism of U_2 and U_1.

(15) Let H be a many sorted function from U_1 into U_2 and let H_1 be a many sorted function from U_2 into U_3. Suppose H is an isomorphism of U_1 and U_2 and H_1 is an isomorphism of U_2 and U_3. Then $H_1 \circ H$ is an isomorphism of U_1 and U_3.

Let us consider S, U_1, U_2. We say that U_1 and U_2 are isomorphic if and only if:

(Def.12) There exists many sorted function from U_1 into U_2 which is an isomorphism of U_1 and U_2.

Next we state three propositions:

(16) U_1 and U_1 are isomorphic.

(17) If U_1 and U_2 are isomorphic, then U_2 and U_1 are isomorphic.

(18) If U_1 and U_2 are isomorphic and U_2 and U_3 are isomorphic, then U_1 and U_3 are isomorphic.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. Let us assume that F is a homomorphism of U_1 into U_2. The functor $\operatorname{Im} F$ yields a strict non-empty subalgebra of U_2 and is defined as follows:

(Def.13) The sorts of $\operatorname{Im} F = F^{\circ}$ (the sorts of U_1).

We now state several propositions:

(19) Let U_2 be a strict non-empty algebra over S and let F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then F is an epimorphism of U_1 onto U_2 if and only if $\mathrm{Im}\, F = U_2$.

(20) Let F be a many sorted function from U_1 into U_2 and let G be a many sorted function from U_1 into $\mathrm{Im}\, F$. Suppose $F = G$ and F is a homomorphism of U_1 into U_2. Then G is an epimorphism of U_1 onto $\mathrm{Im}\, F$.

(21) Let F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then there exists a many sorted function G from U_1 into $\mathrm{Im}\, F$ such that $F = G$ and G is an epimorphism of U_1 onto $\mathrm{Im}\, F$.

(22) Let U_2 be a strict non-empty subalgebra of U_1 and let G be a many sorted function from U_2 into U_1. If $G = \mathrm{id}_{(\text{the sorts of } U_2)}$, then G is a monomorphism of U_2 into U_1.

(23) Let F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then there exists a many sorted function F_1 from U_1 into $\mathrm{Im}\, F$ and there exists a many sorted function F_2 from $\mathrm{Im}\, F$ into U_2 such that F_1 is an epimorphism of U_1 onto $\mathrm{Im}\, F$ and F_2 is a monomorphism of $\mathrm{Im}\, F$ into U_2 and $F = F_2 \circ F_1$.

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