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Many Sorted Quotient Algebra

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Summary. This article introduces the construction of a many sorted quotient algebra. A few preliminary notions such as a many sorted relation, a many sorted equivalence relation, a many sorted congruence and the set of all classes of a many sorted relation are also formulated.

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The notation and terminology used here are introduced in the following papers: [13], [15], [5], [16], [10], [6], [2], [4], [1], [14], [12], [8], [11], [3], [7], and [9].

1. Many Sorted Relation

In this paper S will be a non void non empty many sorted signature and o will be an operation symbol of S.

A function is binary relation yielding if:

(Def.1) For arbitrary x such that $x \in \operatorname{dom} \mathrm{it}$ holds $\mathrm{it}(x)$ is a binary relation.

Let I be a set. Observe that there exists a many sorted set of I which is binary relation yielding.

Let I be a set. A many sorted relation of I is a binary relation yielding many sorted set of I.

Let I be a set and let A, B be many sorted sets of I. A many sorted set of I is said to be a many sorted relation between A and B if:

(Def.2) For arbitrary i such that $i \in I$ holds $\mathrm{it}(i)$ is a relation between $A(i)$ and $B(i)$.

Let I be a set and let A, B be many sorted sets of I. Note that every many sorted relation between A and B is binary relation yielding.

Let I be a set and let A be a many sorted set of I. A many sorted relation of A is a many sorted relation between A and A.

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Let I be a set and let A be a many sorted set of I. A many sorted relation of A is equivalence if it satisfies the condition (Def.3).

(Def.3) Let i be arbitrary and let R be a binary relation on $A(i)$. If $i \in I$ and $\mathrm{it}(i) = R$, then R is an equivalence relation of $A(i)$.

Let I be a non empty set, let A, B be many sorted sets of I, let F be a many sorted relation between A and B, and let i be an element of I. Then $F(i)$ is a relation between $A(i)$ and $B(i)$.

Let S be a non empty many sorted signature and let U_1 be an algebra over S.

(Def.4) A many sorted relation of the sorts of U_1 is said to be a many sorted relation of U_1.

Let S be a non empty many sorted signature and let U_1 be an algebra over S. A many sorted relation of U_1 is equivalence if:

(Def.5) It is equivalence.

Let S be a non void non empty many sorted signature and let U_1 be an algebra over S. Note that there exists a many sorted relation of U_1 which is equivalence.

One can prove the following proposition

(1) Let S be a non void non empty many sorted signature, and let U_1 be an algebra over S, and let R be an equivalence many sorted relation of U_1, and let s be a sort symbol of S. Then $R(s)$ is an equivalence relation of (the sorts of U_1)(s).

Let S be a non void non empty many sorted signature and let U_1 be a non-empty algebra over S. An equivalence many sorted relation of U_1 is called a congruence of U_1 if it satisfies the condition (Def.6).

(Def.6) Let o be an operation symbol of S and let x, y be elements of $\mathrm{Args}(o, U_1)$. Suppose that for every natural number n such that $n \in \mathrm{dom}\, x$ holds $\langle x(n), y(n)\rangle \in \mathrm{it}(\pi_n \mathrm{Arity}(o))$. Then $\langle(\mathrm{Den}(o, U_1))(x), (\mathrm{Den}(o, U_1))(y)\rangle \in$ it(the result sort of o).

Let S be a non void non empty many sorted signature, let U_1 be an algebra over S, let R be an equivalence many sorted relation of U_1, and let i be an element of the carrier of S. Then $R(i)$ is an equivalence relation of (the sorts of U_1)(i).

Let S be a non void non empty many sorted signature, let U_1 be an algebra over S, let R be an equivalence many sorted relation of U_1, let i be an element of the carrier of S, and let x be an element of (the sorts of U_1)(i). The functor $[x]_R$ yields a subset of (the sorts of U_1)(i) and is defined by:

(Def.7) $[x]_R = [x]_{R(i)}$.

Let us consider S, let U_1 be a non-empty algebra over S, and let R be a congruence of U_1. The functor $\mathrm{Classes}\, R$ yields a non-empty many sorted set of the carrier of S and is defined by:

(Def.8) For every element s of the carrier of S holds $(\mathrm{Classes}\, R)(s) = \mathrm{Classes}\, R(s)$.

2. Many Sorted Quotient Algebra

Let us consider S, let M_1, M_2 be many sorted sets of the operation symbols of S, let F be a many sorted function from M_1 into M_2, and let o be an operation symbol of S. Then $F(o)$ is a function from $M_1(o)$ into $M_2(o)$.

Let I be a non empty set, let p be a finite sequence of elements of I, and let X be a non-empty many sorted set of I. Then $X \cdot p$ is a non-empty many sorted set of $\operatorname{dom} p$.

Let us consider S, o, let A be a non-empty algebra over S, let R be a congruence of A, and let x be an element of $\mathrm{Args}(o, A)$. The functor $R\#x$ yields an element of $\prod(\mathrm{Classes}\, R \cdot \mathrm{Arity}(o))$ and is defined as follows:

(Def.9) For every natural number n such that $n \in \operatorname{dom} \mathrm{Arity}(o)$ holds $(R\#x)(n) = [x(n)]_{R(\pi_n \mathrm{Arity}(o))}$.

Let us consider S, o, let A be a non-empty algebra over S, and let R be a congruence of A. The functor $\mathrm{QuotRes}(R, o)$ yielding a function from ((the sorts of A) $\cdot$ (the result sort of S))(o) into (Classes $R \cdot$ (the result sort of S))(o) is defined as follows:

(Def.10) For every element x of (the sorts of A)(the result sort of o) holds $(\mathrm{QuotRes}(R, o))(x) = [x]_R$.

The functor $\mathrm{QuotArgs}(R, o)$ yielding a function from ((the sorts of A)$^\#$ $\cdot$ (the arity of S))(o) into ((Classes R)$^\#$ $\cdot$ (the arity of S))(o) is defined as follows:

(Def.11) For every element x of $\mathrm{Args}(o, A)$ holds $(\mathrm{QuotArgs}(R, o))(x) = R\#x$.

Let us consider S, let A be a non-empty algebra over S, and let R be a congruence of A. The functor $\mathrm{QuotRes}(R)$ yielding a many sorted function from (the sorts of A) $\cdot$ (the result sort of S) into Classes $R \cdot$ (the result sort of S) is defined as follows:

(Def.12) For every operation symbol o of S holds $(\mathrm{QuotRes}(R))(o) = \mathrm{QuotRes}(R, o)$.

The functor $\mathrm{QuotArgs}(R)$ yielding a many sorted function from (the sorts of A)$^\#$ $\cdot$ (the arity of S) into (Classes R)$^\#$ $\cdot$ (the arity of S) is defined as follows:

(Def.13) For every operation symbol o of S holds $(\mathrm{QuotArgs}(R))(o) = \mathrm{QuotArgs}(R, o)$.

Next we state the proposition

(2) Let A be a non-empty algebra over S, and let R be a congruence of A, and let x be arbitrary. Suppose $x \in$ ((Classes R)$^\#$ $\cdot$ (the arity of S))(o). Then there exists an element a of $\mathrm{Args}(o, A)$ such that $x = R\#a$.

Let us consider S, o, let A be a non-empty algebra over S, and let R be a congruence of A. The functor $\mathrm{QuotCharact}(R, o)$ yields a function from ((Classes R)$^\#$ $\cdot$ (the arity of S))(o) into (Classes $R \cdot$ (the result sort of S))(o) and is defined as follows:

(Def.14) For every element a of Args(o, A) such that $R\#a \in ((\text{Classes}\, R)^{\#} \cdot$ (the arity of S))(o) holds $(\text{QuotCharact}(R, o))(R\#a) = (\text{QuotRes}(R, o) \cdot \text{Den}(o, A))(a)$.

Let us consider S, let A be a non-empty algebra over S, and let R be a congruence of A. The functor QuotCharact(R) yielding a many sorted function from $(\text{Classes}\, R)^{\#} \cdot$(the arity of S) into Classes $R \cdot$(the result sort of S) is defined as follows:

(Def.15) For every operation symbol o of S holds $(\text{QuotCharact}(R))(o) = \text{QuotCharact}(R, o)$.

Let us consider S, let U_1 be a non-empty algebra over S, and let R be a congruence of U_1. The functor QuotMSAlg(R) yielding a strict non-empty algebra over S is defined by:

(Def.16) $\text{QuotMSAlg}(R) = \langle \text{Classes}\, R, \text{QuotCharact}(R) \rangle$.

Let us consider S, let U_1 be a non-empty algebra over S, let R be a congruence of U_1, and let s be a sort symbol of S. The functor MSNatHom(U_1, R, s) yielding a function from (the sorts of U_1)(s) into (Classes R)(s) is defined as follows:

(Def.17) For arbitrary x such that $x \in$ (the sorts of U_1)(s) holds $(\text{MSNatHom}(U_1, R, s))(x) = [x]_{R(s)}$.

Let us consider S, let U_1 be a non-empty algebra over S, and let R be a congruence of U_1. The functor MSNatHom(U_1, R) yielding a many sorted function from U_1 into QuotMSAlg(R) is defined by:

(Def.18) For every sort symbol s of S holds $(\text{MSNatHom}(U_1, R))(s) = \text{MSNatHom}(U_1, R, s)$.

Next we state the proposition

(3) Let S be a non void non empty many sorted signature, and let U_1 be a non-empty algebra over S, and let R be a congruence of U_1. Then MSNatHom(U_1, R) is an epimorphism of U_1 onto QuotMSAlg(R).

Let us consider S, let U_1, U_2 be non-empty algebras over S, let F be a many sorted function from U_1 into U_2, and let s be a sort symbol of S. The functor Congruence(F, s) yields an equivalence relation of (the sorts of U_1)(s) and is defined as follows:

(Def.19) For all elements x, y of (the sorts of U_1)(s) holds $\langle x, y \rangle \in \text{Congruence}(F, s)$ iff $F(s)(x) = F(s)(y)$.

Let us consider S, let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. Let us assume that F is a homomorphism of U_1 into U_2. The functor Congruence(F) yielding a congruence of U_1 is defined by:

(Def.20) For every sort symbol s of S holds $(\text{Congruence}(F))(s) = \text{Congruence}(F, s)$.

Let us consider S, let U_1, U_2 be non-empty algebras over S, let F be a many sorted function from U_1 into U_2, and let s be a sort symbol of S. Let us assume that F is a homomorphism of U_1 into U_2. The functor MSHomQuot(F, s) yields a function from (the sorts of QuotMSAlg(Congruence(F)))(s) into (the sorts of U_2)(s) and is defined as follows:

(Def.21) For every element x of (the sorts of U_1)(s) holds (MSHomQuot(F, s)) ($[x]_{\text{Congruence}(F,s)}$) = $F(s)(x)$.

Let us consider S, let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. Let us assume that F is a homomorphism of U_1 into U_2. The functor MSHomQuot(F) yields a many sorted function from QuotMSAlg(Congruence(F)) into U_2 and is defined by:

(Def.22) For every sort symbol s of S holds (MSHomQuot(F))(s) = MSHomQuot(F, s).

The following propositions are true:

(4) Let S be a non void non empty many sorted signature, and let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then MSHomQuot(F) is a monomorphism of QuotMSAlg(Congruence(F)) into U_2.

(5) Let S be a non void non empty many sorted signature, and let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. Suppose F is an epimorphism of U_1 onto U_2. Then MSHomQuot(F) is an isomorphism of QuotMSAlg(Congruence(F)) and U_2.

(6) Let S be a non void non empty many sorted signature, and let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. If F is an epimorphism of U_1 onto U_2, then QuotMSAlg(Congruence(F)) and U_2 are isomorphic.

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