

# A Compiler of Arithmetic Expressions for SCM<sup>1</sup>

Grzegorz Bancerek  
Polish Academy of Sciences  
Institute of Mathematics  
Warsaw

Piotr Rudnicki  
University of Alberta  
Department of Computing Science  
Edmonton

**Summary.** We define a set of binary arithmetic expressions with the following operations:  $+$ ,  $-$ ,  $\cdot$ ,  $\text{mod}$ , and  $\text{div}$  and formalize the common meaning of the expressions in the set of integers. Then, we define a compile function that for a given expression results in a program for the **SCM** machine defined by Nakamura and Trybulec in [13]. We prove that the generated program when loaded into the machine and executed computes the value of the expression. The program uses additional memory and runs in time linear in length of the expression.

MML Identifier: `SCM_COMP`.

The articles [16], [12], [1], [21], [18], [20], [17], [9], [10], [3], [2], [13], [14], [19], [15], [5], [4], [8], [11], [6], and [7] provide the terminology and notation for this paper.

The following two propositions are true:

- (1) Let  $I_1, I_2$  be finite sequences of elements of the instructions of **SCM**, and let  $D$  be a finite sequence of elements of  $\mathbb{Z}$ , and let  $i_1, p_1, d_1$  be natural numbers. Then every state with instruction counter on  $i_1$ , with  $I_1 \frown I_2$  located from  $p_1$ , and  $D$  from  $d_1$  is a state with instruction counter on  $i_1$ , with  $I_1$  located from  $p_1$ , and  $D$  from  $d_1$  and a state with instruction counter on  $i_1$ , with  $I_2$  located from  $p_1 + \text{len } I_1$ , and  $D$  from  $d_1$ .
- (2) Let  $I_1, I_2$  be finite sequences of elements of the instructions of **SCM**, and let  $i_1, p_1, d_1, k, i_2$  be natural numbers, and let  $s$  be a state with instruction counter on  $i_1$ , with  $I_1 \frown I_2$  located from  $p_1$ , and  $\varepsilon_{\mathbb{Z}}$  from  $d_1$ , and let  $u$  be a state of **SCM**. Suppose  $u = (\text{Computation}(s))(k)$  and

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<sup>1</sup>This work was partially supported by NSERC Grant OGP9207 while the first author visited University of Alberta, May-June 1993.

$\mathbf{i}_{(i_2)} = \mathbf{IC}_u$ . Then  $u$  is a state with instruction counter on  $i_2$ , with  $I_2$  located from  $p_1 + \text{len } I_1$ , and  $\varepsilon_z$  from  $d_1$ .

The binary strict non empty tree construction structure  $\text{AE}_{\text{SCM}}$  with terminals, nonterminals, and useful nonterminals is defined by the conditions (Def.1).

- (Def.1) (i) The terminals of  $\text{AE}_{\text{SCM}} = \text{Data-Loc}_{\text{SCM}}$ ,  
(ii) the nonterminals of  $\text{AE}_{\text{SCM}} = \{1, 5\}$ , and  
(iii) for all symbols  $x, y, z$  of  $\text{AE}_{\text{SCM}}$  holds  $x \Rightarrow \langle y, z \rangle$  iff  $x \in \{1, 5\}$ .

A binary term is an element of  $\text{TS}(\text{AE}_{\text{SCM}})$ .

Let  $n_1$  be a nonterminal of  $\text{AE}_{\text{SCM}}$  and let  $t_1, t_2$  be binary terms. Then  $n_1\text{-tree}(t_1, t_2)$  is a binary term.

Let  $t$  be a terminal of  $\text{AE}_{\text{SCM}}$ . Then the root tree of  $t$  is a binary term.

Let  $t$  be a terminal of  $\text{AE}_{\text{SCM}}$ . The functor  ${}^@t$  yielding a data-location is defined as follows:

- (Def.2)  ${}^@t = t$ .

One can prove the following propositions:

- (3) For every nonterminal  $n_1$  of  $\text{AE}_{\text{SCM}}$  holds  $n_1 = \langle 0, 0 \rangle$  or  $n_1 = \langle 0, 1 \rangle$  or  $n_1 = \langle 0, 2 \rangle$  or  $n_1 = \langle 0, 3 \rangle$  or  $n_1 = \langle 0, 4 \rangle$ .  
(4) (i)  $\langle 0, 0 \rangle$  is a nonterminal of  $\text{AE}_{\text{SCM}}$ ,  
(ii)  $\langle 0, 1 \rangle$  is a nonterminal of  $\text{AE}_{\text{SCM}}$ ,  
(iii)  $\langle 0, 2 \rangle$  is a nonterminal of  $\text{AE}_{\text{SCM}}$ ,  
(iv)  $\langle 0, 3 \rangle$  is a nonterminal of  $\text{AE}_{\text{SCM}}$ , and  
(v)  $\langle 0, 4 \rangle$  is a nonterminal of  $\text{AE}_{\text{SCM}}$ .

Let  $t_3, t_4$  be binary terms. The functor  $t_3 + t_4$  yields a binary term and is defined as follows:

- (Def.3)  $t_3 + t_4 = \langle 0, 0 \rangle\text{-tree}(t_3, t_4)$ .

The functor  $t_3 - t_4$  yielding a binary term is defined as follows:

- (Def.4)  $t_3 - t_4 = \langle 0, 1 \rangle\text{-tree}(t_3, t_4)$ .

The functor  $t_3 \cdot t_4$  yields a binary term and is defined by:

- (Def.5)  $t_3 \cdot t_4 = \langle 0, 2 \rangle\text{-tree}(t_3, t_4)$ .

The functor  $t_3 \div t_4$  yields a binary term and is defined by:

- (Def.6)  $t_3 \div t_4 = \langle 0, 3 \rangle\text{-tree}(t_3, t_4)$ .

The functor  $t_3 \bmod t_4$  yielding a binary term is defined as follows:

- (Def.7)  $t_3 \bmod t_4 = \langle 0, 4 \rangle\text{-tree}(t_3, t_4)$ .

We now state the proposition

- (5) Let  $t_5$  be a binary term. Then  
(i) there exists a terminal  $t$  of  $\text{AE}_{\text{SCM}}$  such that  $t_5 = \text{the root tree of } t$ , or  
(ii) there exist binary terms  $t_1, t_2$  such that  $t_5 = t_1 + t_2$  or  $t_5 = t_1 - t_2$  or  $t_5 = t_1 \cdot t_2$  or  $t_5 = t_1 \div t_2$  or  $t_5 = t_1 \bmod t_2$ .

Let  $o$  be a nonterminal of  $\text{AE}_{\text{SCM}}$  and let  $i, j$  be integers. The functor  $o(i, j)$  yielding an integer is defined as follows:

- (Def.8) (i)  $o(i, j) = i + j$  if  $o = \langle 0, 0 \rangle$ ,  
(ii)  $o(i, j) = i - j$  if  $o = \langle 0, 1 \rangle$ ,  
(iii)  $o(i, j) = i \cdot j$  if  $o = \langle 0, 2 \rangle$ ,  
(iv)  $o(i, j) = i \div j$  if  $o = \langle 0, 3 \rangle$ ,  
(v)  $o(i, j) = i \bmod j$  if  $o = \langle 0, 4 \rangle$ .

Let  $s$  be a state of **SCM** and let  $t$  be a terminal of  $\text{AE}_{\text{SCM}}$ . Then  $s(t)$  is an integer.

$\mathbb{Z}$  is a non empty subset of  $\mathbb{R}$ .

One can verify that every element of  $\mathbb{Z}$  is integer.

Let  $D$  be a non empty set, let  $f$  be a function from  $\mathbb{Z}$  into  $D$ , and let  $x$  be an integer. Then  $f(x)$  is an element of  $D$ .

Let  $s$  be a state of **SCM** and let  $t_5$  be a binary term. The functor  $t_5^{\textcircled{a}} s$  yields an integer and is defined by the condition (Def.9).

- (Def.9) There exists a function  $f$  from  $\text{TS}(\text{AE}_{\text{SCM}})$  into  $\mathbb{Z}$  such that
- (i)  $t_5^{\textcircled{a}} s = f(t_5)$ ,
  - (ii) for every terminal  $t$  of  $\text{AE}_{\text{SCM}}$  holds  $f(\text{the root tree of } t) = s(t)$ , and
  - (iii) for every nonterminal  $n_1$  of  $\text{AE}_{\text{SCM}}$  and for all binary terms  $t_1, t_2$  and for all symbols  $r_1, r_2$  of  $\text{AE}_{\text{SCM}}$  such that  $r_1 = \text{the root label of } t_1$  and  $r_2 = \text{the root label of } t_2$  and  $n_1 \Rightarrow \langle r_1, r_2 \rangle$  and for all elements  $x_1, x_2$  of  $\mathbb{Z}$  such that  $x_1 = f(t_1)$  and  $x_2 = f(t_2)$  holds  $f(n_1\text{-tree}(t_1, t_2)) = n_1(x_1, x_2)$ .

One can prove the following three propositions:

- (6) For every state  $s$  of **SCM** and for every terminal  $t$  of  $\text{AE}_{\text{SCM}}$  holds (the root tree of  $t$ ) $^{\textcircled{a}} s = s(t)$ .
- (7) For every state  $s$  of **SCM** and for every nonterminal  $n_1$  of  $\text{AE}_{\text{SCM}}$  and for all binary terms  $t_1, t_2$  holds  $(n_1\text{-tree}(t_1, t_2))^{\textcircled{a}} s = n_1(t_1^{\textcircled{a}} s, t_2^{\textcircled{a}} s)$ .
- (8) Let  $s$  be a state of **SCM** and let  $t_1, t_2$  be binary terms. Then  $(t_1 + t_2)^{\textcircled{a}} s = (t_1^{\textcircled{a}} s) + (t_2^{\textcircled{a}} s)$  and  $(t_1 - t_2)^{\textcircled{a}} s = (t_1^{\textcircled{a}} s) - (t_2^{\textcircled{a}} s)$  and  $t_1 \cdot t_2^{\textcircled{a}} s = (t_1^{\textcircled{a}} s) \cdot (t_2^{\textcircled{a}} s)$  and  $(t_1 \div t_2)^{\textcircled{a}} s = (t_1^{\textcircled{a}} s) \div (t_2^{\textcircled{a}} s)$  and  $(t_1 \bmod t_2)^{\textcircled{a}} s = (t_1^{\textcircled{a}} s) \bmod (t_2^{\textcircled{a}} s)$ .

Let  $n_1$  be a nonterminal of  $\text{AE}_{\text{SCM}}$  and let  $n$  be a natural number. The functor  $\text{Selfwork}(n_1, n)$  yielding an element of (the instructions of **SCM qua set**) $^*$  is defined as follows:

- (Def.10) (i)  $\text{Selfwork}(n_1, n) = \langle \text{AddTo}(\mathbf{d}_n, \mathbf{d}_{n+1}) \rangle$  if  $n_1 = \langle 0, 0 \rangle$ ,  
(ii)  $\text{Selfwork}(n_1, n) = \langle \text{SubFrom}(\mathbf{d}_n, \mathbf{d}_{n+1}) \rangle$  if  $n_1 = \langle 0, 1 \rangle$ ,  
(iii)  $\text{Selfwork}(n_1, n) = \langle \text{MultBy}(\mathbf{d}_n, \mathbf{d}_{n+1}) \rangle$  if  $n_1 = \langle 0, 2 \rangle$ ,  
(iv)  $\text{Selfwork}(n_1, n) = \langle \text{Divide}(\mathbf{d}_n, \mathbf{d}_{n+1}) \rangle$  if  $n_1 = \langle 0, 3 \rangle$ ,  
(v)  $\text{Selfwork}(n_1, n) = \langle \text{Divide}(\mathbf{d}_n, \mathbf{d}_{n+1}), \mathbf{d}_n := \mathbf{d}_{n+1} \rangle$  if  $n_1 = \langle 0, 4 \rangle$ .

Let  $t_5$  be a binary term and let  $a_1$  be a natural number. The functor  $\text{Compile}(t_5, a_1)$  yielding a finite sequence of elements of the instructions of **SCM** is defined by the condition (Def.11).

- (Def.11) There exists a function  $f$  from  $\text{TS}(\text{AE}_{\text{SCM}})$  into ((the instructions of **SCM qua set**) $^*$ ) $^{\mathbb{N}}$  such that
- (i)  $\text{Compile}(t_5, a_1) = (f(t_5) \text{ qua element of ((the instructions of **SCM**$

- $\mathbf{qua\ set}^*)^{\mathbb{N}}(a_1)$ ,
- (ii) for every terminal  $t$  of  $\mathbf{AE}_{\mathbf{SCM}}$  there exists a function  $g$  from  $\mathbb{N}$  into (the instructions of  $\mathbf{SCM\ qua\ set}^*$ ) such that  $g = f(\text{the root tree of } t)$  and for every natural number  $n$  holds  $g(n) = \langle \mathbf{d}_n := @t \rangle$ , and
  - (iii) for every nonterminal  $n_1$  of  $\mathbf{AE}_{\mathbf{SCM}}$  and for all binary terms  $t_3, t_4$  and for all symbols  $r_1, r_2$  of  $\mathbf{AE}_{\mathbf{SCM}}$  such that  $r_1 = \text{the root label of } t_3$  and  $r_2 = \text{the root label of } t_4$  and  $n_1 \Rightarrow \langle r_1, r_2 \rangle$  there exist functions  $g, f_1, f_2$  from  $\mathbb{N}$  into (the instructions of  $\mathbf{SCM\ qua\ set}^*$ ) such that  $g = f(n_1\text{-tree}(t_3, t_4))$  and  $f_1 = f(t_3)$  and  $f_2 = f(t_4)$  and for every natural number  $n$  holds  $g(n) = f_1(n) \wedge f_2(n+1) \wedge \text{Selfwork}(n_1, n)$ .

One can prove the following propositions:

- (9) For every terminal  $t$  of  $\mathbf{AE}_{\mathbf{SCM}}$  and for every natural number  $n$  holds  $\text{Compile}(\text{the root tree of } t, n) = \langle \mathbf{d}_n := @t \rangle$ .
- (10) Let  $n_1$  be a nonterminal of  $\mathbf{AE}_{\mathbf{SCM}}$ , and let  $t_3, t_4$  be binary terms, and let  $n$  be a natural number, and let  $r_1, r_2$  be symbols of  $\mathbf{AE}_{\mathbf{SCM}}$ . Suppose  $r_1 = \text{the root label of } t_3$  and  $r_2 = \text{the root label of } t_4$  and  $n_1 \Rightarrow \langle r_1, r_2 \rangle$ . Then  $\text{Compile}(n_1\text{-tree}(t_3, t_4), n) = (\text{Compile}(t_3, n)) \wedge \text{Compile}(t_4, n+1) \wedge \text{Selfwork}(n_1, n)$ .

Let  $t$  be a terminal of  $\mathbf{AE}_{\mathbf{SCM}}$ . The functor  $\mathbf{d}^{-1}(t)$  yielding a natural number is defined as follows:

(Def.12)  $\mathbf{d}_{\mathbf{d}^{-1}(t)} = t$ .

Let  $n_2, n_3$  be natural numbers. Then  $\max(n_2, n_3)$  is a natural number.

Let  $t_5$  be a binary term. The functor  $\max_{\mathbf{DL}}(t_5)$  yielding a natural number is defined by the condition (Def.13).

- (Def.13) There exists a function  $f$  from  $\mathbf{TS}(\mathbf{AE}_{\mathbf{SCM}})$  into  $\mathbb{N}$  such that
- (i)  $\max_{\mathbf{DL}}(t_5) = f(t_5)$ ,
  - (ii) for every terminal  $t$  of  $\mathbf{AE}_{\mathbf{SCM}}$  holds  $f(\text{the root tree of } t) = \mathbf{d}^{-1}(t)$ , and
  - (iii) for every nonterminal  $n_1$  of  $\mathbf{AE}_{\mathbf{SCM}}$  and for all binary terms  $t_1, t_2$  and for all symbols  $r_1, r_2$  of  $\mathbf{AE}_{\mathbf{SCM}}$  such that  $r_1 = \text{the root label of } t_1$  and  $r_2 = \text{the root label of } t_2$  and  $n_1 \Rightarrow \langle r_1, r_2 \rangle$  and for all natural numbers  $x_1, x_2$  such that  $x_1 = f(t_1)$  and  $x_2 = f(t_2)$  holds  $f(n_1\text{-tree}(t_1, t_2)) = \max(x_1, x_2)$ .

One can prove the following propositions:

- (11) For every terminal  $t$  of  $\mathbf{AE}_{\mathbf{SCM}}$  holds  $\max_{\mathbf{DL}}(\text{the root tree of } t) = \mathbf{d}^{-1}(t)$ .
- (12) For every nonterminal  $n_1$  of  $\mathbf{AE}_{\mathbf{SCM}}$  and for all binary terms  $t_1, t_2$  holds  $\max_{\mathbf{DL}}(n_1\text{-tree}(t_1, t_2)) = \max(\max_{\mathbf{DL}}(t_1), \max_{\mathbf{DL}}(t_2))$ .
- (13) Let  $t_5$  be a binary term and let  $s_1, s_2$  be states of  $\mathbf{SCM}$ . Suppose that for every natural number  $d_2$  such that  $d_2 \leq \max_{\mathbf{DL}}(t_5)$  holds  $s_1(\mathbf{d}_{(d_2)}) = s_2(\mathbf{d}_{(d_2)})$ . Then  $t_5^@ s_1 = t_5^@ s_2$ .
- (14) Let  $t_5$  be a binary term, and let  $a_1, n, k$  be natural numbers, and let  $s$  be a state with instruction counter on  $n$ , with  $\text{Compile}(t_5, a_1)$  located

from  $n$ , and  $\varepsilon_Z$  from  $k$ . Suppose  $a_1 > \max_{\text{DL}}(t_5)$ . Then there exists a natural number  $i$  and there exists a state  $u$  of **SCM** such that

- (i)  $u = (\text{Computation}(s))(i + 1)$ ,
  - (ii)  $i + 1 = \text{len Compile}(t_5, a_1)$ ,
  - (iii)  $\mathbf{IC}_{(\text{Computation}(s))(i)} = \mathbf{i}_{n+i}$ ,
  - (iv)  $\mathbf{IC}_u = \mathbf{i}_{n+(i+1)}$ ,
  - (v)  $u(\mathbf{d}_{(a_1)}) = t_5^{\textcircled{a}} s$ , and
  - (vi) for every natural number  $d_2$  such that  $d_2 < a_1$  holds  $s(\mathbf{d}_{(d_2)}) = u(\mathbf{d}_{(d_2)})$ .
- (15) Let  $t_5$  be a binary term, and let  $a_1, n, k$  be natural numbers, and let  $s$  be a state with instruction counter on  $n$ , with  $(\text{Compile}(t_5, a_1)) \hat{\sim} \langle \mathbf{halt}_{\text{SCM}} \rangle$  located from  $n$ , and  $\varepsilon_Z$  from  $k$ . Suppose  $a_1 > \max_{\text{DL}}(t_5)$ . Then  $s$  is halting and  $(\text{Result}(s))(\mathbf{d}_{(a_1)}) = t_5^{\textcircled{a}} s$  and the complexity of  $s = \text{len Compile}(t_5, a_1)$ .

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*Received December 30, 1993*

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