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On Kolmogorov Topological Spaces [1]

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Summary. Let X be a topological space. X is said to be T_0*-space* (or *Kolmogorov space*) provided for every pair of distinct points x, $y \in X$ there exists an open subset of X containing exactly one of these points; equivalently, for every pair of distinct points x, $y \in X$ there exists a closed subset of X containing exactly one of these points (see [1], [6], [2]).

The purpose is to list some of the standard facts on Kolmogorov spaces, using Mizar formalism. As a sample we formulate the following characteristics of such spaces: *X is a Kolmogorov space iff for every pair of distinct points x, $y \in X$ the closures $\overline{\{x\}}$ and $\overline{\{y\}}$ are distinct.*

There is also reviewed analogous facts on Kolmogorov subspaces of topological spaces. In the presented approach T_0-subsets are introduced and some of their properties developed.

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The articles [10], [11], [9], [7], [8], [5], [4], and [3] provide the terminology and notation for this paper.

1. Subspaces

Let Y be a non empty topological structure. We see that the subspace of Y is a non empty topological structure and it can be characterized by the following (equivalent) condition:

(Def.1) (i) The carrier of it $\subseteq$ the carrier of Y, and

(ii) for every subset G_0 of it holds G_0 is open iff there exists a subset G of Y such that G is open and $G_0 = G \cap$ (the carrier of it).

Next we state two propositions:

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(1) Let Y be a non empty topological structure, and let Y_0 be a subspace of Y, and let G_0 be a subset of Y_0. Then G_0 is open if and only if there exists a subset G of Y such that G is open and $G_0 = G \cap$ (the carrier of Y_0).

(2) Let Y be a non empty topological structure, and let Y_0 be a subspace of Y, and let G be a subset of Y. Suppose G is open. Then there exists a subset G_0 of Y_0 such that G_0 is open and $G_0 = G \cap$ (the carrier of Y_0).

Let Y be a non empty topological structure. We see that the subspace of Y is a non empty topological structure and it can be characterized by the following (equivalent) condition:

(Def.2) (i) The carrier of it $\subseteq$ the carrier of Y, and

(ii) for every subset F_0 of it holds F_0 is closed iff there exists a subset F of Y such that F is closed and $F_0 = F \cap$ (the carrier of it).

We now state two propositions:

(3) Let Y be a non empty topological structure, and let Y_0 be a subspace of Y, and let F_0 be a subset of Y_0. Then F_0 is closed if and only if there exists a subset F of Y such that F is closed and $F_0 = F \cap$ (the carrier of Y_0).

(4) Let Y be a non empty topological structure, and let Y_0 be a subspace of Y, and let F be a subset of Y. Suppose F is closed. Then there exists a subset F_0 of Y_0 such that F_0 is closed and $F_0 = F \cap$ (the carrier of Y_0).

2. Kolmogorov Spaces

A topological structure is T_0 if it satisfies the condition (Def.3).

(Def.3) Let x, y be points of it. Suppose $x \neq y$. Then

(i) there exists a subset V of it such that V is open and $x \in V$ and $y \notin V$, or

(ii) there exists a subset W of it such that W is open and $x \notin W$ and $y \in W$.

Let us observe that a non empty topological structure is T_0 if it satisfies the condition (Def.4).

(Def.4) Let x, y be points of it. Suppose $x \neq y$. Then

(i) there exists a subset E of it such that E is closed and $x \in E$ and $y \notin E$, or

(ii) there exists a subset F of it such that F is closed and $x \notin F$ and $y \in F$.

Let us mention that every non empty topological structure which is trivial is also T_0 and every non empty topological structure which is non T_0 is also non trivial.

One can verify that there exists a topological space which is strict T_0 and non empty and there exists a topological space which is strict non T_0 and non empty.

One can check the following observations:

* every topological space which is discrete is also T_0,
* every topological space which is non T_0 is also non discrete,
* every topological space which is anti-discrete and non trivial is also non T_0,
* every topological space which is anti-discrete and T_0 is also trivial, and
* every topological space which is T_0 and non trivial is also non anti-discrete.

Let us observe that a topological space is T_0 if:

(Def.5) For all points x, y of it such that $x \neq y$ holds $\overline{\{x\}} \neq \overline{\{y\}}$.

Let us observe that a topological space is T_0 if:

(Def.6) For all points x, y of it such that $x \neq y$ holds $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$.

Let us observe that a topological space is T_0 if:

(Def.7) For all points x, y of it such that $x \neq y$ and $x \in \overline{\{y\}}$ holds $\overline{\{y\}} \not\subseteq \overline{\{x\}}$.

One can verify the following observations:

* every topological space which is almost discrete and T_0 is also discrete,
* every topological space which is almost discrete and non discrete is also non T_0, and
* every topological space which is non discrete and T_0 is also non almost discrete.

A Kolmogorov space is a T_0 topological space. A non-Kolmogorov space is a non T_0 topological space.

Let us observe that there exists a Kolmogorov space which is non trivial and strict and there exists a non-Kolmogorov space which is non trivial and strict.

3. T_0-Subsets

Let Y be a topological structure. A subset of Y is T_0 if it satisfies the condition (Def.8).

(Def.8) Let x, y be points of Y. Suppose $x \in$ it and $y \in$ it and $x \neq y$. Then there exists a subset V of Y such that V is open and $x \in V$ and $y \notin V$ or there exists a subset W of Y such that W is open and $x \notin W$ and $y \in W$.

Let Y be a non empty topological structure. Let us observe that a subset of Y is T_0 if it satisfies the condition (Def.9).

(Def.9) Let x, y be points of Y. Suppose $x \in$ it and $y \in$ it and $x \neq y$. Then

(i) there exists a subset E of Y such that E is closed and $x \in E$ and $y \notin E$, or

(ii) there exists a subset F of Y such that F is closed and $x \notin F$ and $y \in F$.

Next we state two propositions:

(5) Let Y_0, Y_1 be topological structures, and let D_0 be a subset of Y_0, and let D_1 be a subset of Y_1. Suppose the topological structure of Y_0 = the topological structure of Y_1 and $D_0 = D_1$. If D_0 is T_0, then D_1 is T_0.

(6) Let Y be a non empty topological structure and let A be a subset of Y. Suppose A = the carrier of Y. Then A is T_0 if and only if Y is T_0.

In the sequel Y will denote a non empty topological structure.

The following propositions are true:

(7) For all subsets A, B of Y such that $B \subseteq A$ holds if A is T_0, then B is T_0.

(8) For all subsets A, B of Y such that A is T_0 or B is T_0 holds $A \cap B$ is T_0.

(9) Let A, B be subsets of Y. Suppose A is open or B is open. If A is T_0 and B is T_0, then $A \cup B$ is T_0.

(10) Let A, B be subsets of Y. Suppose A is closed or B is closed. If A is T_0 and B is T_0, then $A \cup B$ is T_0.

(11) For every subset A of Y such that A is discrete holds A is T_0.

(12) For every non empty subset A of Y such that A is anti-discrete and A is not trivial holds A is not T_0.

Let X be a topological space. Let us observe that a subset of X is T_0 if:

(Def.10) For all points x, y of X such that $x \in$ it and $y \in$ it and $x \neq y$ holds $\overline{\{x\}} \neq \overline{\{y\}}$.

Let X be a topological space. Let us observe that a subset of X is T_0 if:

(Def.11) For all points x, y of X such that $x \in$ it and $y \in$ it and $x \neq y$ holds $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$.

Let X be a topological space. Let us observe that a subset of X is T_0 if:

(Def.12) For all points x, y of X such that $x \in$ it and $y \in$ it and $x \neq y$ holds if $x \in \overline{\{y\}}$, then $\overline{\{y\}} \not\subseteq \overline{\{x\}}$.

In the sequel X will denote a topological space.

The following two propositions are true:

(13) Every empty subset of X is T_0.

(14) For every point x of X holds $\{x\}$ is T_0.

4. Kolmogorov Subspaces

Let Y be a non empty topological structure. Observe that there exists a subspace of Y which is strict and T_0.

Let Y be a non empty topological structure. Let us observe that a subspace of Y is T_0 if it satisfies the condition (Def.13).

(Def.13) Let x, y be points of Y. Suppose x is a point of it and y is a point of it and $x \neq y$. Then there exists a subset V of Y such that V is open and

$x \in V$ and $y \notin V$ or there exists a subset W of Y such that W is open and $x \notin W$ and $y \in W$.

Let Y be a non empty topological structure. Let us observe that a subspace of Y is T_0 if it satisfies the condition (Def.14).

(Def.14) Let x, y be points of Y. Suppose x is a point of it and y is a point of it and $x \neq y$. Then

(i) there exists a subset E of Y such that E is closed and $x \in E$ and $y \notin E$, or

(ii) there exists a subset F of Y such that F is closed and $x \notin F$ and $y \in F$.

In the sequel Y denotes a non empty topological structure.

The following propositions are true:

(15) Let Y_0 be a subspace of Y and let A be a subset of Y. Suppose $A =$ the carrier of Y_0. Then A is T_0 if and only if Y_0 is T_0.

(16) Let Y_0 be a subspace of Y and let Y_1 be a T_0 subspace of Y. If Y_0 is a subspace of Y_1, then Y_0 is T_0.

Let X be a topological space. One can check that there exists a subspace of X which is strict and T_0.

In the sequel X is a topological space.

The following propositions are true:

(17) For every T_0 subspace X_1 of X and for every subspace X_2 of X such that X_1 meets X_2 holds $X_1 \cap X_2$ is T_0.

(18) For all T_0 subspaces X_1, X_2 of X such that X_1 is open or X_2 is open holds $X_1 \cup X_2$ is T_0.

(19) For all T_0 subspaces X_1, X_2 of X such that X_1 is closed or X_2 is closed holds $X_1 \cup X_2$ is T_0.

Let X be a topological space. A Kolmogorov subspace of X is a T_0 subspace of X.

Next we state the proposition

(20) Let X be a topological space and let A_0 be a non empty subset of X. Suppose A_0 is T_0. Then there exists a strict Kolmogorov subspace X_0 of X such that $A_0 =$ the carrier of X_0.

Let X be a non trivial topological space. One can verify that there exists a Kolmogorov subspace of X which is proper and strict.

Let X be a Kolmogorov space. Observe that every subspace of X is T_0.

Let X be a non-Kolmogorov space. One can check that every subspace of X which is non proper is also non T_0 and every subspace of X which is T_0 is also proper.

Let X be a non-Kolmogorov space. Note that there exists a subspace of X which is strict and non T_0.

Let X be a non-Kolmogorov space. A non-Kolmogorov subspace of X is a non T_0 subspace of X.

We now state the proposition

(21) Let X be a non-Kolmogorov space and let A_0 be a subset of X. Suppose A_0 is not T_0. Then there exists a strict non-Kolmogorov subspace X_0 of X such that $A_0 =$ the carrier of X_0.

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