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Categories without Uniqueness of cod and dom

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Summary. Category theory had been formalized in Mizar quite early [8]. This had been done closely to the handbook of S. McLane [11]. In this paper we use a different approach. Category is a triple

$$\langle O, \{\langle o_1, o_2\rangle\}_{o_1,o_2 \in O}, \{\circ_{o_1,o_2,o_3}\}_{o_1,o_2,o_3 \in O}\rangle$$

where $\circ_{o_1,o_2,o_3} : \langle o_2, o_3\rangle \times \langle o_1, o_2\rangle \to \langle o_1, o_3\rangle$ that satisfies usual conditions (associativity and the existence of the identities). This approach is closer to the way in which categories are presented in homological algebra (e.g. [1], pp.58-59). We do not assume that $\langle o_1, o_2\rangle$'s are mutually disjoint. If f is simultaneously a morphism from o_1 to o_2 and o'_1 to o_2 ($o_1 \neq o'_1$) than different compositions are used ($\circ_{o_1,o_2,o_3}$ or $\circ_{o'_1,o_2,o_3}$) to compose it with a morphism g from o_2 to o_3. The operation $g \cdot f$ has actually six arguments (two visible and four hidden: three objects and the category).

We introduce some simple properties of categories. Perhaps more than necessary. It is partially caused by the formalization. The functional categories are characterized by the following properties:

- quasi-functional that means that morphisms are functions (rather meaningless, if it stands alone)
- semi-functional that means that the composition of morphism is the composition of functions, provided they are functions.
- pseudo-functional that means that the composition of morphisms is the composition of functions.

For categories pseudo-functional is just quasi-functional and semi-functional, but we work in a bit more general setting. Similarly the concept of a discrete category is split into two:

- quasi-discrete that means that $\langle o_1, o_2\rangle$ is empty for $o_1 \neq o_2$ and
- pseudo-discrete that means that $\langle o, o\rangle$ is trivial, i.e. consists of the identity only, in a category.

We plan to follow Semadeni-Wiweger book [14], in the development the category theory in Mizar. However, the beginning is not very close to [14], because of the approach adopted and because we work in Tarski-Grothendieck set theory.

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The terminology and notation used in this paper have been introduced in the following articles: [19], [21], [20], [15], [22], [2], [6], [7], [3], [13], [5], [10], [4], [16], [9], [18], [12], and [17].

1. Preliminaries

One can prove the following proposition

(1) For every non empty set A and for all sets B, C, D such that $[: A, B :] \subseteq [: C, D :]$ or $[: B, A :] \subseteq [: D, C :]$ holds $B \subseteq D$.

In the sequel i, j, k, x are arbitrary.

Let A be a functional set. Observe that every subset of A is functional.

Let f be a function yielding function and let C be a set. Observe that $f \restriction C$ is function yielding.

Let f be a function. One can verify that $\{f\}$ is functional.

Next we state four propositions:

(2) For every set A holds $\mathrm{id}_A \in A^A$.

(3) $\emptyset^\emptyset = \{\mathrm{id}_\emptyset\}$.

(4) For all sets A, B, C and for all functions f, g such that $f \in B^A$ and $g \in C^B$ holds $g \cdot f \in C^A$.

(5) For all sets A, B, C such that $B^A \neq \emptyset$ and $C^B \neq \emptyset$ holds $C^A \neq \emptyset$.

Let A, B be sets. One can check that B^A is functional.

We now state two propositions:

(6) For all sets A, B and for every function f such that $f \in B^A$ holds $\mathrm{dom}\, f = A$ and $\mathrm{rng}\, f \subseteq B$.

(7) Let A, B be sets, and let F be a many sorted set indexed by $[: B, A :]$, and let C be a subset of A, and let D be a subset of B, and let x, y be arbitrary. If $x \in C$ and $y \in D$, then $F(y, x) = (F \restriction [: D, C :])(y, x)$.

In this article we present several logical schemes. The scheme *MSSLambdaD* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

> There exists a many sorted set M indexed by $\mathcal{A}$ such that for every element i of $\mathcal{A}$ holds $M(i) = \mathcal{F}(i)$

for all values of the parameters.

The scheme *MSSLambda2* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a binary functor $\mathcal{F}$ yielding arbitrary, and states that:

> There exists a many sorted set M indexed by $[: \mathcal{A}, \mathcal{B} :]$ such that for all i, j such that $i \in \mathcal{A}$ and $j \in \mathcal{B}$ holds $M(i, j) = \mathcal{F}(i, j)$

for all values of the parameters.

The scheme *MSSLambda2D* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$ and a binary functor $\mathcal{F}$ yielding arbitrary, and states that:

> There exists a many sorted set M indexed by $[:\mathcal{A}, \mathcal{B}:]$ such that for every element i of $\mathcal{A}$ and for every element j of $\mathcal{B}$ holds $M(i, j) = \mathcal{F}(i, j)$

for all values of the parameters.

The scheme *MSSLambda3* concerns sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a ternary functor $\mathcal{F}$ yielding arbitrary, and states that:

> There exists a many sorted set M indexed by $[:\mathcal{A}, \mathcal{B}, \mathcal{C}:]$ such that for all i, j, k such that $i \in \mathcal{A}$ and $j \in \mathcal{B}$ and $k \in \mathcal{C}$ holds $M(i, j, k) = \mathcal{F}(i, j, k)$

for all values of the parameters.

The scheme *MSSLambda3D* deals with non empty sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and a ternary functor $\mathcal{F}$ yielding arbitrary, and states that:

> There exists a many sorted set M indexed by $[:\mathcal{A}, \mathcal{B}, \mathcal{C}:]$ such that for every element i of $\mathcal{A}$ and for every element j of $\mathcal{B}$ and for every element k of $\mathcal{C}$ holds $M(i, j, k) = \mathcal{F}(i, j, k)$

for all values of the parameters.

One can prove the following propositions:

(8) Let A, B be sets and let N, M be many sorted sets indexed by $[:A, B:]$. If for all i, j such that $i \in A$ and $j \in B$ holds $N(i, j) = M(i, j)$, then $M = N$.

(9) Let A, B be non empty sets and let N, M be many sorted sets indexed by $[:A, B:]$. Suppose that for every element i of A and for every element j of B holds $N(i, j) = M(i, j)$. Then $M = N$.

(10) Let A be a set and let N, M be many sorted sets indexed by $[:A, A, A:]$. Suppose that for all i, j, k such that $i \in A$ and $j \in A$ and $k \in A$ holds $N(i, j, k) = M(i, j, k)$. Then $M = N$.

(11) $[\langle i, j \rangle \mapsto k] = \langle i, j \rangle \longmapsto k$.

(12) $[\langle i, j \rangle \mapsto k](i, j) = k$.

2. Graphs

We consider graphs as extensions of 1-sorted structure as systems

$\langle$ a carrier, arrows $\rangle$,

where the carrier is a set and the arrows constitute a many sorted set indexed by $[:$the carrier, the carrier$:]$.

Let G be a graph.

(Def.1) An element of the carrier of G is called an object of G.

Let G be a graph and let o_1, o_2 be objects of G. The functor $\langle o_1, o_2 \rangle$ is defined as follows:

(Def.2) $\langle o_1, o_2 \rangle = (\text{the arrows of } G)(o_1, o_2)$.

Let G be a graph and let o_1, o_2 be objects of G.

(Def.3) An element of $\langle o_1, o_2 \rangle$ is said to be a morphism from o_1 to o_2.

Let G be a graph. We say that G is transitive if and only if:

(Def.4) For all objects o_1, o_2, o_3 of G such that $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ holds $\langle o_1, o_3 \rangle \neq \emptyset$.

3. Many Sorted Binary Compositions

Let I be a set and let G be a many sorted set indexed by $[\!: I, I :\!]$. The functor $\{\!|G|\!\}$ yields a many sorted set indexed by $[\!: I, I, I :\!]$ and is defined as follows:

(Def.5) For all i, j, k such that $i \in I$ and $j \in I$ and $k \in I$ holds $(\{\!|G|\!\})(i, j, k) = G(i, k)$.

Let H be a many sorted set indexed by $[\!: I, I :\!]$. The functor $\{\!|G, H|\!\}$ yielding a many sorted set indexed by $[\!: I, I, I :\!]$ is defined by:

(Def.6) For all i, j, k such that $i \in I$ and $j \in I$ and $k \in I$ holds $(\{\!|G, H|\!\})(i, j, k) = [\!: H(j, k), G(i, j) :\!]$.

Let I be a set and let G be a many sorted set indexed by $[\!: I, I :\!]$. A binary composition of G is a many sorted function from $\{\!|G, G|\!\}$ into $\{\!|G|\!\}$.

Let I be a non empty set, let G be a many sorted set indexed by $[\!: I, I :\!]$, let o be a binary composition of G, and let i, j, k be elements of I. Then $o(i, j, k)$ is a function from $[\!: G(j, k), G(i, j) :\!]$ into $G(i, k)$.

Let I be a non empty set and let G be a many sorted set indexed by $[\!: I, I :\!]$. A binary composition of G is associative if it satisfies the condition (Def.7).

(Def.7) Let i, j, k, l be elements of I and let f, g, h be arbitrary. Suppose $f \in G(i, j)$ and $g \in G(j, k)$ and $h \in G(k, l)$. Then $\mathrm{it}(i, k, l)(h, \mathrm{it}(i, j, k)(g, f)) = \mathrm{it}(i, j, l)(\mathrm{it}(j, k, l)(h, g), f)$.

A binary composition of G has right units if it satisfies the condition (Def.8).

(Def.8) Let i be an element of I. Then there exists arbitrary e such that $e \in G(i, i)$ and for every element j of I and for arbitrary f such that $f \in G(i, j)$ holds $\mathrm{it}(i, i, j)(f, e) = f$.

A binary composition of G has left units if it satisfies the condition (Def.9).

(Def.9) Let j be an element of I. Then there exists arbitrary e such that $e \in G(j, j)$ and for every element i of I and for arbitrary f such that $f \in G(i, j)$ holds $\mathrm{it}(i, j, j)(e, f) = f$.

4. Categories

We introduce category structures which are extensions of graph and are systems

$\langle$ a carrier, arrows, a composition $\rangle$, where the carrier is a set, the arrows constitute a many sorted set indexed by $[\!:$ the carrier, the carrier $:\!]$, and the composition is a binary composition of the arrows.

Let us observe that there exists a category structure which is strict and non empty.

Let C be a non empty category structure and let o_1, o_2, o_3 be objects of C. Let us assume that $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_1, o_3 \rangle \neq \emptyset$. Let f be a morphism from o_1 to o_2 and let g be a morphism from o_2 to o_3. The functor $g \cdot f$ yields a morphism from o_1 to o_3 and is defined by:

(Def.10) $g \cdot f =$ (the composition of C)$(o_1, o_2, o_3)(g, f)$.

A function is compositional if:

(Def.11) If $x \in \operatorname{dom}$ it, then there exist functions f, g such that $x = \langle\!\langle g, f \rangle\!\rangle$ and $\operatorname{it}(x) = g \cdot f$.

Let A, B be functional sets. Observe that there exists a many sorted function of $[\!: A, B :\!]$ which is compositional.

Next we state the proposition

(13) Let A, B be functional sets, and let F be a compositional many sorted set indexed by $[\!: A, B :\!]$, and let g, f be functions. If $g \in A$ and $f \in B$, then $F(g, f) = g \cdot f$.

Let A, B be functional sets.

(Def.12) $\operatorname{FuncComp}(A, B)$ is a compositional many sorted function of $[\!: B, A :\!]$.

The following propositions are true:

(14) For all sets A, B, C holds $\operatorname{rng} \operatorname{FuncComp}(B^A, C^B) \subseteq C^A$.

(15) For every set o holds $\operatorname{FuncComp}(\{\mathrm{id}_o\}, \{\mathrm{id}_o\}) = [\langle \mathrm{id}_o, \mathrm{id}_o \rangle \mapsto \mathrm{id}_o]$.

(16) For all functional sets A, B and for every subset A_1 of A and for every subset B_1 of B holds $\operatorname{FuncComp}(A_1, B_1) = \operatorname{FuncComp}(A, B) \restriction [\!: B_1, A_1 :\!]$.

Let C be a non empty category structure. We say that C is quasi-functional if and only if:

(Def.13) For all objects a_1, a_2 of C holds $\langle a_1, a_2 \rangle \subseteq {a_2}^{a_1}$.

We say that C is semi-functional if and only if the condition (Def.14) is satisfied.

(Def.14) Let a_1, a_2, a_3 be objects of C. Suppose $\langle a_1, a_2 \rangle \neq \emptyset$ and $\langle a_2, a_3 \rangle \neq \emptyset$ and $\langle a_1, a_3 \rangle \neq \emptyset$. Let f be a morphism from a_1 to a_2, and let g be a morphism from a_2 to a_3, and let f', g' be functions. If $f = f'$ and $g = g'$, then $g \cdot f = g' \cdot f'$.

We say that C is pseudo-functional if and only if:

(Def.15) For all objects o_1, o_2, o_3 of C holds (the composition of C)$(o_1, o_2, o_3) = \operatorname{FuncComp}({o_2}^{o_1}, {o_3}^{o_2}) \restriction [\!: \langle o_2, o_3 \rangle, \langle o_1, o_2 \rangle :\!]$.

Let X be a non empty set, let A be a many sorted set indexed by $[\!: X, X :\!]$, and let C be a binary composition of A. Note that $\langle X, A, C \rangle$ is non empty.

Let us observe that there exists a non empty category structure which is strict and pseudo-functional.

One can prove the following propositions:

(17) Let C be a non empty category structure and let a_1, a_2, a_3 be objects of C. Suppose if $\langle a_1, a_3 \rangle = \emptyset$, then $\langle a_1, a_2 \rangle = \emptyset$ or $\langle a_2, a_3 \rangle = \emptyset$. Then (the composition of C)(a_1, a_2, a_3) is a function from $[\!: \langle a_2, a_3 \rangle, \langle a_1, a_2 \rangle :\!]$ into $\langle a_1, a_3 \rangle$.

(18) Let C be a pseudo-functional non empty category structure and let a_1, a_2, a_3 be objects of C. Suppose $\langle a_1, a_2 \rangle \neq \emptyset$ and $\langle a_2, a_3 \rangle \neq \emptyset$ and $\langle a_1, a_3 \rangle \neq \emptyset$. Let f be a morphism from a_1 to a_2, and let g be a morphism from a_2 to a_3, and let f', g' be functions. If $f = f'$ and $g = g'$, then $g \cdot f = g' \cdot f'$.

Let A be a non empty set. The functor Ens_A yielding a strict pseudo-functional non empty category structure is defined as follows:

(Def.16) The carrier of $\mathrm{Ens}_A = A$ and for all objects a_1, a_2 of Ens_A holds $\langle a_1, a_2 \rangle = {a_2}^{a_1}$.

Let C be a non empty category structure. We say that C is associative if and only if:

(Def.17) The composition of C is associative.

We say that C has units if and only if:

(Def.18) The composition of C has left units and right units.

Let us mention that there exists a non empty category structure which is transitive associative and strict and has units.

The following propositions are true:

(19) Let C be a transitive non empty category structure and let a_1, a_2, a_3 be objects of C. Then (the composition of C)(a_1, a_2, a_3) is a function from $[\!: \langle a_2, a_3 \rangle, \langle a_1, a_2 \rangle :\!]$ into $\langle a_1, a_3 \rangle$.

(20) Let C be a transitive non empty category structure and let a_1, a_2, a_3 be objects of C. Then dom (the composition of C)$(a_1, a_2, a_3) = [\!: \langle a_2, a_3 \rangle, \langle a_1, a_2 \rangle :\!]$ and rng (the composition of C)$(a_1, a_2, a_3) \subseteq \langle a_1, a_3 \rangle$.

(21) For every non empty category structure C with units and for every object o of C holds $\langle o, o \rangle \neq \emptyset$.

Let A be a non empty set. Observe that Ens_A is transitive and associative and has units.

Let us mention that every non empty category structure which is quasi-functional semi-functional and transitive is also pseudo-functional and every non empty category structure which is pseudo-functional and transitive and has units is also quasi-functional and semi-functional.

A category is a transitive associative non empty category structure with units.

5. Identities

One can prove the following proposition

(22) Let C be a transitive non empty category structure and let o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$. Let f be a morphism from o_1 to o_2 and let g be a morphism from o_2 to o_3. Then $g \cdot f =$ (the composition of C)$(o_1, o_2, o_3)(g, f)$.

Let C be a non empty category structure with units and let o be an object of C. The functor id_o yielding a morphism from o to o is defined by:

(Def.19) For every object o' of C such that $\langle o, o' \rangle \neq \emptyset$ and for every morphism a from o to o' holds $a \cdot \mathrm{id}_o = a$.

One can prove the following three propositions:

(23) For every non empty category structure C with units and for every object o of C holds $\mathrm{id}_o \in \langle o, o \rangle$.

(24) Let C be a non empty category structure with units and let o_1, o_2 be objects of C. If $\langle o_1, o_2 \rangle \neq \emptyset$, then for every morphism a from o_1 to o_2 holds $\mathrm{id}_{(o_2)} \cdot a = a$.

(25) Let C be an associative transitive non empty category structure and let o_1, o_2, o_3, o_4 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_3, o_4 \rangle \neq \emptyset$. Let a be a morphism from o_1 to o_2, and let b be a morphism from o_2 to o_3, and let c be a morphism from o_3 to o_4. Then $c \cdot (b \cdot a) = (c \cdot b) \cdot a$.

6. Discrete categories

Let C be a category structure. We say that C is quasi-discrete if and only if:

(Def.20) For all objects i, j of C such that $\langle i, j \rangle \neq \emptyset$ holds $i = j$.

We say that C is pseudo-discrete if and only if:

(Def.21) For every object i of C holds $\langle i, i \rangle$ is trivial.

One can prove the following proposition

(26) Let C be a non empty category structure with units. Then C is pseudo-discrete if and only if for every object o of C holds $\langle o, o \rangle = \{\mathrm{id}_o\}$.

Let us observe that every category structure which is trivial is also quasi-discrete.

One can prove the following proposition

(27) Ens_1 is pseudo-discrete and trivial.

Let us note that there exists a category which is pseudo-discrete trivial and strict.

Let us observe that there exists a category which is quasi-discrete pseudo-discrete trivial and strict.

A discrete category is a quasi-discrete pseudo-discrete category.

Let A be a non empty set. The functor $\mathrm{DiscrCat}(A)$ yields a quasi-discrete strict non empty category structure and is defined by:

(Def.22) The carrier of DiscrCat(A) $= A$ and for every object i of DiscrCat(A) holds $\langle i, i\rangle = \{\mathrm{id}_i\}$.

One can verify that every category structure which is quasi-discrete is also transitive.

One can prove the following propositions:

(28) Let A be a non empty set and let o_1, o_2, o_3 be objects of DiscrCat(A). If $o_1 \neq o_2$ or $o_2 \neq o_3$, then (the composition of DiscrCat(A))(o_1, o_2, o_3) $= \emptyset$.

(29) For every non empty set A and for every object o of DiscrCat(A) holds (the composition of DiscrCat(A))(o, o, o) $= [\langle \mathrm{id}_o, \mathrm{id}_o\rangle \mapsto \mathrm{id}_o]$.

Let A be a non empty set. Note that DiscrCat(A) is pseudo-functional pseudo-discrete and associative and has units.

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