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On the Group of Automorphisms of Universal Algebra & Many Sorted Algebra

Artur Korniłowicz
Warsaw University
Białystok

Summary. The aim of the article is to check the compatibility of the automorphisms of universal algebras introduced in [8] and the corresponding concept for many sorted algebras introduced in [9].

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The notation and terminology used in this paper have been introduced in the following articles: [2], [17], [20], [21], [5], [6], [4], [14], [16], [11], [13], [18], [19], [1], [10], [3], [8], [12], [15], [9], and [7].

1. On the Group of Automorphisms of Universal Algebra

In this paper U_1 denotes a universal algebra and f, g denote functions from U_1 into U_1.

One can prove the following proposition

(1) $\mathrm{id}_{(\text{the carrier of } U_1)}$ is an isomorphism of U_1 and U_1.

Let us consider U_1. The functor $\mathrm{UAAut}(U_1)$ yields a non empty set of functions from the carrier of U_1 to the carrier of U_1 and is defined by the conditions (Def.1).

(Def.1) (i) Every element of $\mathrm{UAAut}(U_1)$ is a function from U_1 into U_1, and
(ii) for every function h from U_1 into U_1 holds $h \in \mathrm{UAAut}(U_1)$ iff h is an isomorphism of U_1 and U_1.

Next we state several propositions:

(2) $\mathrm{UAAut}(U_1) \subseteq (\text{the carrier of } U_1)^{\text{the carrier of } U_1}$.

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(3) For every f holds $f \in \mathrm{UAAut}(U_1)$ iff f is an isomorphism of U_1 and U_1.

(4) $\mathrm{id}_{(\text{the carrier of } U_1)} \in \mathrm{UAAut}(U_1)$.

(5) For all f, g such that f is an element of $\mathrm{UAAut}(U_1)$ and $g = f^{-1}$ holds g is an isomorphism of U_1 and U_1.

(6) For every element f of $\mathrm{UAAut}(U_1)$ holds $f^{-1} \in \mathrm{UAAut}(U_1)$.

(7) For all elements f_1, f_2 of $\mathrm{UAAut}(U_1)$ holds $f_1 \cdot f_2 \in \mathrm{UAAut}(U_1)$.

Let us consider U_1. The functor $\mathrm{UAAutComp}(U_1)$ yields a binary operation on $\mathrm{UAAut}(U_1)$ and is defined as follows:

(Def.2) For all elements x, y of $\mathrm{UAAut}(U_1)$ holds $(\mathrm{UAAutComp}(U_1))(x, y) = y \cdot x$.

Let us consider U_1. The functor $\mathrm{UAAutGroup}(U_1)$ yielding a group is defined by:

(Def.3) $\mathrm{UAAutGroup}(U_1) = \langle \mathrm{UAAut}(U_1), \mathrm{UAAutComp}(U_1) \rangle$.

Let us consider U_1. Note that $\mathrm{UAAutGroup}(U_1)$ is strict.

The following propositions are true:

(8) Let x, y be elements of the carrier of $\mathrm{UAAutGroup}(U_1)$ and let f, g be elements of $\mathrm{UAAut}(U_1)$. If $x = f$ and $y = g$, then $x \cdot y = g \cdot f$.

(9) $\mathrm{id}_{(\text{the carrier of } U_1)} = 1_{\mathrm{UAAutGroup}(U_1)}$.

(10) For every element f of $\mathrm{UAAut}(U_1)$ and for every element g of the carrier of $\mathrm{UAAutGroup}(U_1)$ such that $f = g$ holds $f^{-1} = g^{-1}$.

2. Some Properties of Many Sorted Functions

In the sequel I is a set and A, B, C are many sorted sets indexed by I.

Let us consider I, A, B. We say that A is transformable to B if and only if:

(Def.4) For arbitrary i such that $i \in I$ holds if $B(i) = \emptyset$, then $A(i) = \emptyset$.

Let us observe that the predicate introduced above is reflexive.

Next we state several propositions:

(11) If A is transformable to B and B is transformable to C, then A is transformable to C.

(12) For arbitrary x and for every many sorted set A indexed by $\{x\}$ holds $A = \{x\} \longmapsto A(x)$.

(13) For all function yielding functions F, G, H holds $(H \circ G) \circ F = H \circ (G \circ F)$.

(14) Let A, B be non-empty many sorted sets indexed by I and let F be a many sorted function from A into B. If F is "1-1" and "onto", then F^{-1} is "1-1" and "onto".

(15) Let A, B be non-empty many sorted sets indexed by I and let F be a many sorted function from A into B. If F is "1-1" and "onto", then $(F^{-1})^{-1} = F$.

(16) For all function yielding functions F, G such that F is "1-1" and G is "1-1" holds $G \circ F$ is "1-1".

(17) Let B, C be non-empty many sorted sets indexed by I, and let F be a many sorted function from A into B, and let G be a many sorted function from B into C. If F is "onto" and G is "onto", then $G \circ F$ is "onto".

(18) Let A, B, C be non-empty many sorted sets indexed by I, and let F be a many sorted function from A into B, and let G be a many sorted function from B into C. Suppose F is "1-1" and "onto" and G is "1-1" and "onto". Then $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

(19) Let A, B be non-empty many sorted sets indexed by I, and let F be a many sorted function from A into B, and let G be a many sorted function from B into A. If F is "1-1" and "onto" and $G \circ F = \mathrm{id}_A$, then $G = F^{-1}$.

3. On the Group of Automorphisms of Many Sorted Algebra

In the sequel S will be a non void non empty many sorted signature and U_2, U_3 will be non-empty algebras over S.

Let us consider I, A, B. The functor MSFuncs(A, B) yields a many sorted set indexed by I and is defined as follows:

(Def.5) For arbitrary i such that $i \in I$ holds $(\mathrm{MSFuncs}(A, B))(i) = B(i)^{A(i)}$.

One can prove the following propositions:

(20) Let h be a many sorted set indexed by I. If $h = \mathrm{MSFuncs}(A, B)$, then for arbitrary i such that $i \in I$ holds $h(i) = B(i)^{A(i)}$.

(21) Let A, B be many sorted sets indexed by I. Suppose A is transformable to B. Let x be arbitrary. If $x \in \prod \mathrm{MSFuncs}(A, B)$, then x is a many sorted function from A into B.

(22) Let A, B be many sorted sets indexed by I. Suppose A is transformable to B. Let g be a many sorted function from A into B. Then $g \in \prod \mathrm{MSFuncs}(A, B)$.

(23) For all many sorted sets A, B indexed by I such that A is transformable to B holds MSFuncs(A, B) is non-empty.

Let us consider I, A, B. Let us assume that A is transformable to B. A non empty set is said to be a set of manysorted functions from A into B if:

(Def.6) For arbitrary x such that $x \in$ it holds x is a many sorted function from A into B.

Let us consider I, A. Note that MSFuncs(A, A) is non-empty.

Let us consider S, U_2, U_3. A set of manysorted functions from U_2 into U_3 is a set of manysorted functions from the sorts of U_2 into the sorts of U_3.

Let I be a set and let D be a many sorted set indexed by I. Note that there exists a set of manysorted functions from D into D which is non empty.

We now state four propositions:

(24) id_A is "onto".

(25) id_A is "1-1".

(26) $\mathrm{id}_{(\text{the sorts of } U_2)}$ is an isomorphism of U_2 and U_2.

(27) $\mathrm{id}_{(\text{the sorts of } U_2)} \in \prod$ MSFuncs(the sorts of U_2, the sorts of U_2).

Let us consider S, U_2. The functor MSAAut(U_2) yielding a set of manysorted functions from the sorts of U_2 into the sorts of U_2 is defined by the conditions (Def.7).

(Def.7) (i) Every element of MSAAut(U_2) is a many sorted function from U_2 into U_2, and

(ii) for every many sorted function h from U_2 into U_2 holds $h \in$ MSAAut(U_2) iff h is an isomorphism of U_2 and U_2.

One can prove the following propositions:

(28) For every many sorted function F from U_2 into U_2 holds $F \in$ MSAAut(U_2) iff F is an isomorphism of U_2 and U_2.

(29) For every element f of MSAAut(U_2) holds $f \in \prod$ MSFuncs(the sorts of U_2, the sorts of U_2).

(30) MSAAut(U_2) $\subseteq \prod$ MSFuncs(the sorts of U_2, the sorts of U_2).

(31) $\mathrm{id}_{(\text{the sorts of } U_2)} \in$ MSAAut(U_2).

(32) For every element f of MSAAut(U_2) holds $f^{-1} \in$ MSAAut(U_2).

(33) For all elements f_1, f_2 of MSAAut(U_2) holds $f_1 \circ f_2 \in$ MSAAut(U_2).

(34) For every many sorted function F from MSAlg(U_1) into MSAlg(U_1) and for every element f of UAAut(U_1) such that $F = \{0\} \longmapsto f$ holds $F \in$ MSAAut(MSAlg(U_1)).

Let us consider S, U_2. The functor MSAAutComp(U_2) yields a binary operation on MSAAut(U_2) and is defined as follows:

(Def.8) For all elements x, y of MSAAut(U_2) holds (MSAAutComp(U_2))(x, y) $= y \circ x$.

Let us consider S, U_2. The functor MSAAutGroup(U_2) yields a group and is defined by:

(Def.9) MSAAutGroup(U_2) $= \langle$MSAAut(U_2), MSAAutComp(U_2)$\rangle$.

Let us consider S, U_2. Observe that MSAAutGroup(U_2) is strict.

The following three propositions are true:

(35) Let x, y be elements of the carrier of MSAAutGroup(U_2) and let f, g be elements of MSAAut(U_2). If $x = f$ and $y = g$, then $x \cdot y = g \circ f$.

(36) $\mathrm{id}_{(\text{the sorts of } U_2)} = 1_{\mathrm{MSAAutGroup}(U_2)}$.

(37) For every element f of MSAAut(U_2) and for every element g of MSAAutGroup(U_2) such that $f = g$ holds $f^{-1} = g^{-1}$.

4. On the Relationship of Automorphisms of 1-sorted and Many Sorted Algebras

Next we state several propositions:

(38) Let U_4, U_5 be universal algebras. Suppose U_4 and U_5 are similar. Let F be a many sorted function from $\mathrm{MSAlg}(U_4)$ into $(\mathrm{MSAlg}(U_5)\,\mathrm{over}\,\mathrm{MSSign}(U_4))$. Then $F(0)$ is a function from U_4 into U_5.

(39) For every element f of $\mathrm{UAAut}(U_1)$ holds $\{0\} \longmapsto f$ is a many sorted function from $\mathrm{MSAlg}(U_1)$ into $\mathrm{MSAlg}(U_1)$.

(40) Let h be a function. Suppose $\mathrm{dom}\, h = \mathrm{UAAut}(U_1)$ and for arbitrary x such that $x \in \mathrm{UAAut}(U_1)$ holds $h(x) = \{0\} \longmapsto x$. Then h is a homomorphism from $\mathrm{UAAutGroup}(U_1)$ to $\mathrm{MSAAutGroup}(\mathrm{MSAlg}(U_1))$.

(41) Let h be a homomorphism from $\mathrm{UAAutGroup}(U_1)$ to $\mathrm{MSAAutGroup}(\mathrm{MSAlg}(U_1))$. Suppose that for arbitrary x such that $x \in \mathrm{UAAut}(U_1)$ holds $h(x) = \{0\} \longmapsto x$. Then h is an isomorphism.

(42) $\mathrm{UAAutGroup}(U_1)$ and $\mathrm{MSAAutGroup}(\mathrm{MSAlg}(U_1))$ are isomorphic.

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