FORMALIZED MATHEMATICS
Volume 5, Number 2, 1996
Warsaw University - Białystok

Combining of Circuits [1]

Yatsuka Nakamura
Shinshu University, Nagano

Grzegorz Bancerek
Institute of Mathematics
Polish Academy of Sciences

Summary. We continue the formalisation of circuits started in [15,14,13,12]. Our goal was to work out the notation of combining circuits which could be employed to prove the properties of real circuits.

MML Identifier: `CIRCCOMB`.

The terminology and notation used in this paper are introduced in the following papers: [20], [23], [21], [25], [5], [3], [4], [9], [6], [16], [8], [7], [17], [22], [1], [2], [24], [10], [19], [11], [18], [15], [14], [13], and [12].

1. Combining of Many Sorted Signatures

Let S be a many sorted signature. A gate of S is an element of the operation symbols of S.

Let A be a set and let X be a set. Then $A \longmapsto X$ is a many sorted set indexed by A.

Let A be a set and let X be a non empty set. One can check that $A \longmapsto X$ is non-empty.

Let A be a set and let f be a function. One can verify that $A \longmapsto f$ is function yielding.

Let f, g be non-empty functions. Note that $f +\cdot g$ is non-empty.

Let A, B be sets, let f be a many sorted set indexed by A, and let g be a many sorted set indexed by B. Then $f +\cdot g$ is a many sorted set indexed by $A \cup B$.

We now state several propositions:

[1]This work was written while the second author visited Shinshu University, July–August 1994.

ISSN 1426–2630

(1) For all functions f_1, f_2, g_1, g_2 such that $\operatorname{rng} g_1 \subseteq \operatorname{dom} f_1$ and $\operatorname{rng} g_2 \subseteq \operatorname{dom} f_2$ and $f_1 \approx f_2$ holds $(f_1 +\cdot f_2) \cdot (g_1 +\cdot g_2) = f_1 \cdot g_1 +\cdot f_2 \cdot g_2$.

(2) For all functions f_1, f_2, g such that $\operatorname{rng} g \subseteq \operatorname{dom} f_1$ and $\operatorname{rng} g \subseteq \operatorname{dom} f_2$ and $f_1 \approx f_2$ holds $f_1 \cdot g = f_2 \cdot g$.

(3) Let A, B be sets, and let f be a many sorted set indexed by A, and let g be a many sorted set indexed by B. If $f \subseteq g$, then $f^\# \subseteq g^\#$.

(4) For all sets X, Y, x, y holds $X \longmapsto x \approx Y \longmapsto y$ iff $x = y$ or X misses Y.

(5) For all functions f, g, h such that $f \approx g$ and $g \approx h$ and $h \approx f$ holds $f +\cdot g \approx h$.

(6) For every set X and for every non empty set Y and for every finite sequence p of elements of X holds $(X \longmapsto Y)^\#(p) = Y^{\operatorname{len} p}$.

Let A be a set, let f_1, g_1 be non-empty many sorted sets indexed by A, let B be a set, let f_2, g_2 be non-empty many sorted sets indexed by B, let h_1 be a many sorted function from f_1 into g_1, and let h_2 be a many sorted function from f_2 into g_2. Then $h_1 +\cdot h_2$ is a many sorted function from $f_1 +\cdot f_2$ into $g_1 +\cdot g_2$.

Let S_1, S_2 be many sorted signatures. The predicate $S_1 \approx S_2$ is defined by:

(Def.1) The arity of $S_1 \approx$ the arity of S_2 and the result sort of $S_1 \approx$ the result sort of S_2.

Let us notice that this predicate is reflexive and symmetric.

Let S_1, S_2 be non empty many sorted signatures. The functor $S_1 +\cdot S_2$ yielding a strict non empty many sorted signature is defined by the conditions (Def.2).

(Def.2) (i) The carrier of $S_1 +\cdot S_2 =$ (the carrier of S_1) $\cup$ (the carrier of S_2),

(ii) the operation symbols of $S_1 +\cdot S_2 =$ (the operation symbols of S_1) $\cup$ (the operation symbols of S_2),

(iii) the arity of $S_1 +\cdot S_2 =$ (the arity of S_1) $+\cdot$ (the arity of S_2), and

(iv) the result sort of $S_1 +\cdot S_2 =$ (the result sort of S_1) $+\cdot$ (the result sort of S_2).

The following propositions are true:

(7) For all non empty many sorted signatures S_1, S_2, S_3 such that $S_1 \approx S_2$ and $S_2 \approx S_3$ and $S_3 \approx S_1$ holds $S_1 +\cdot S_2 \approx S_3$.

(8) For every non empty many sorted signature S holds $S +\cdot S =$ the many sorted signature of S.

(9) For all non empty many sorted signatures S_1, S_2 such that $S_1 \approx S_2$ holds $S_1 +\cdot S_2 = S_2 +\cdot S_1$.

(10) For all non empty many sorted signatures S_1, S_2, S_3 holds $(S_1 +\cdot S_2) +\cdot S_3 = S_1 +\cdot (S_2 +\cdot S_3)$.

One can verify that there exists a function which is one-to-one.

Next we state four propositions:

(11) Let f be an one-to-one function and let S_1, S_2 be circuit-like non empty many sorted signatures. Suppose the result sort of $S_1 \subseteq f$ and the result

sort of $S_2 \subseteq f$. Then $S_1 +\cdot S_2$ is circuit-like.

(12) For all circuit-like non empty many sorted signatures S_1, S_2 such that InnerVertices(S_1) misses InnerVertices(S_2) holds $S_1 +\cdot S_2$ is circuit-like.

(13) For all non empty many sorted signatures S_1, S_2 such that S_1 is not void or S_2 is not void holds $S_1 +\cdot S_2$ is non void.

(14) For all finite non empty many sorted signatures S_1, S_2 holds $S_1 +\cdot S_2$ is finite.

Let S_1 be a non void non empty many sorted signature and let S_2 be a non empty many sorted signature. Observe that $S_1 +\cdot S_2$ is non void and $S_2 +\cdot S_1$ is non void.

We now state several propositions:

(15) For all non empty many sorted signatures S_1, S_2 such that $S_1 \approx S_2$ holds InnerVertices($S_1 +\cdot S_2$) $=$ InnerVertices(S_1) $\cup$ InnerVertices(S_2) and InputVertices($S_1 +\cdot S_2$) $\subseteq$ InputVertices(S_1) $\cup$ InputVertices(S_2).

(16) For all non empty many sorted signatures S_1, S_2 and for every vertex v_2 of S_2 such that $v_2 \in$ InputVertices($S_1 +\cdot S_2$) holds $v_2 \in$ InputVertices(S_2).

(17) Let S_1, S_2 be non empty many sorted signatures. If $S_1 \approx S_2$, then for every vertex v_1 of S_1 such that $v_1 \in$ InputVertices($S_1 +\cdot S_2$) holds $v_1 \in$ InputVertices(S_1).

(18) Let S_1 be a non empty many sorted signature, and let S_2 be a non void non empty many sorted signature, and let o_2 be an operation symbol of S_2, and let o be an operation symbol of $S_1 +\cdot S_2$. Suppose $o_2 = o$. Then Arity(o) $=$ Arity(o_2) and the result sort of $o =$ the result sort of o_2.

(19) Let S_1 be a non empty many sorted signature and let S_2, S be circuit-like non void non empty many sorted signatures. Suppose $S = S_1 +\cdot S_2$. Let v_2 be a vertex of S_2. Suppose $v_2 \in$ InnerVertices(S_2). Let v be a vertex of S. If $v_2 = v$, then $v \in$ InnerVertices(S) and the action at $v =$ the action at v_2.

(20) Let S_1 be a non void non empty many sorted signature and let S_2 be a non empty many sorted signature. Suppose $S_1 \approx S_2$. Let o_1 be an operation symbol of S_1 and let o be an operation symbol of $S_1 +\cdot S_2$. Suppose $o_1 = o$. Then Arity(o) $=$ Arity(o_1) and the result sort of $o =$ the result sort of o_1.

(21) Let S_1, S be circuit-like non void non empty many sorted signatures and let S_2 be a non empty many sorted signature. Suppose $S_1 \approx S_2$ and $S = S_1 +\cdot S_2$. Let v_1 be a vertex of S_1. Suppose $v_1 \in$ InnerVertices(S_1). Let v be a vertex of S. If $v_1 = v$, then $v \in$ InnerVertices(S) and the action at $v =$ the action at v_1.

2. Combinig of Circuits

Let S_1, S_2 be non empty many sorted signatures, let A_1 be an algebra over S_1, and let A_2 be an algebra over S_2. The predicate $A_1 \approx A_2$ is defined by:

(Def.3) $S_1 \approx S_2$ and the sorts of $A_1 \approx$ the sorts of A_2 and the characteristics of $A_1 \approx$ the characteristics of A_2.

Let S_1, S_2 be non empty many sorted signatures, let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2. Let us assume that the sorts of $A_1 \approx$ the sorts of A_2. The functor $A_1 +\cdot A_2$ yields a strict non-empty algebra over $S_1 +\cdot S_2$ and is defined by the conditions (Def.4).

(Def.4) (i) The sorts of $A_1 +\cdot A_2 =$ (the sorts of A_1) $+\cdot$ (the sorts of A_2), and

(ii) the characteristics of $A_1 +\cdot A_2 =$ (the characteristics of A_1) $+\cdot$ (the characteristics of A_2).

The following propositions are true:

(22) For every non void non empty many sorted signature S and for every algebra A over S holds $A \approx A$.

(23) Let S_1, S_2 be non void non empty many sorted signatures, and let A_1 be an algebra over S_1, and let A_2 be an algebra over S_2. If $A_1 \approx A_2$, then $A_2 \approx A_1$.

(24) Let S_1, S_2, S_3 be non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2, and let A_3 be an algebra over S_3. If $A_1 \approx A_2$ and $A_2 \approx A_3$ and $A_3 \approx A_1$, then $A_1 +\cdot A_2 \approx A_3$.

(25) Let S be a strict non empty many sorted signature and let A be a non-empty algebra over S. Then $A +\cdot A =$ the algebra of A.

(26) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2. If $A_1 \approx A_2$, then $A_1 +\cdot A_2 = A_2 +\cdot A_1$.

(27) Let S_1, S_2, S_3 be non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2, and let A_3 be a non-empty algebra over S_3. Suppose that

(i) the sorts of $A_1 \approx$ the sorts of A_2,

(ii) the sorts of $A_2 \approx$ the sorts of A_3, and

(iii) the sorts of $A_3 \approx$ the sorts of A_1.

Then $(A_1 +\cdot A_2) +\cdot A_3 = A_1 +\cdot (A_2 +\cdot A_3)$.

(28) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be a locally-finite non-empty algebra over S_1, and let A_2 be a locally-finite non-empty algebra over S_2. If the sorts of $A_1 \approx$ the sorts of A_2, then $A_1 +\cdot A_2$ is locally-finite.

(29) For all non-empty functions f, g and for every element x of $\prod f$ and for every element y of $\prod g$ holds $x +\cdot y \in \prod(f +\cdot g)$.

(30) For all non-empty functions f, g and for every element x of $\prod(f +\cdot g)$ holds $x \restriction \operatorname{dom} g \in \prod g$.

(31) For all non-empty functions f, g such that $f \approx g$ and for every element x of $\prod(f +\cdot g)$ holds $x \restriction \operatorname{dom} f \in \prod f$.

(32) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let s_1 be an element of $\prod$ (the sorts of A_1), and let A_2 be a non-empty algebra over S_2, and let s_2 be an element of $\prod$ (the sorts of A_2). If the sorts of $A_1 \approx$ the sorts of A_2, then $s_1 +\cdot s_2 \in \prod$ (the sorts of $A_1 +\cdot A_2$).

(33) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2. Suppose the sorts of $A_1 \approx$ the sorts of A_2. Let s be an element of $\prod$ (the sorts of $A_1 +\cdot A_2$). Then $s \restriction$ (the carrier of S_1) $\in \prod$ (the sorts of A_1) and $s \restriction$ (the carrier of S_2) $\in \prod$ (the sorts of A_2).

(34) Let S_1, S_2 be non void non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2. Suppose the sorts of $A_1 \approx$ the sorts of A_2. Let o be an operation symbol of $S_1 +\cdot S_2$ and let o_2 be an operation symbol of S_2. If $o = o_2$, then $\operatorname{Den}(o, A_1 +\cdot A_2) = \operatorname{Den}(o_2, A_2)$.

(35) Let S_1, S_2 be non void non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2. Suppose the sorts of $A_1 \approx$ the sorts of A_2 and the characteristics of $A_1 \approx$ the characteristics of A_2. Let o be an operation symbol of $S_1 +\cdot S_2$ and let o_1 be an operation symbol of S_1. If $o = o_1$, then $\operatorname{Den}(o, A_1 +\cdot A_2) = \operatorname{Den}(o_1, A_1)$.

(36) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $S = S_1 +\cdot S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S, and let s be a state of A, and let s_2 be a state of A_2. Suppose $s_2 = s \restriction$ (the carrier of S_2). Let g be a gate of S and let g_2 be a gate of S_2. If $g = g_2$, then g depends-on-in $s = g_2$ depends-on-in s_2.

(37) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $S = S_1 +\cdot S_2$ and $S_1 \approx S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S, and let s be a state of A, and let s_1 be a state of A_1. Suppose $s_1 = s \restriction$ (the carrier of S_1). Let g be a gate of S and let g_1 be a gate of S_1. If $g = g_1$, then g depends-on-in $s = g_1$ depends-on-in s_1.

(38) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $S = S_1 +\cdot S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let s be a state of A and let v be a vertex of S. Then

(i) for every state s_1 of A_1 such that $s_1 = s \restriction$ (the carrier of S_1) holds if

$v \in \mathrm{InnerVertices}(S_1)$ or $v \in$ the carrier of S_1 and $v \in \mathrm{InputVertices}(S)$, then $(\mathrm{Following}(s))(v) = (\mathrm{Following}(s_1))(v)$, and

(ii) for every state s_2 of A_2 such that $s_2 = s \upharpoonright$ (the carrier of S_2) holds if $v \in \mathrm{InnerVertices}(S_2)$ or $v \in$ the carrier of S_2 and $v \in \mathrm{InputVertices}(S)$, then $(\mathrm{Following}(s))(v) = (\mathrm{Following}(s_2))(v)$.

(39) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $\mathrm{InnerVertices}(S_1)$ misses $\mathrm{InputVertices}(S_2)$ and $S = S_1 {+\cdot} S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1 {+\cdot} A_2$. Let s be a state of A, and let s_1 be a state of A_1, and let s_2 be a state of A_2. Suppose $s_1 = s \upharpoonright$ (the carrier of S_1) and $s_2 = s \upharpoonright$ (the carrier of S_2). Then $\mathrm{Following}(s) = \mathrm{Following}(s_1) {+\cdot} \mathrm{Following}(s_2)$.

(40) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $\mathrm{InnerVertices}(S_2)$ misses $\mathrm{InputVertices}(S_1)$ and $S = S_1 {+\cdot} S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1 {+\cdot} A_2$. Let s be a state of A, and let s_1 be a state of A_1, and let s_2 be a state of A_2. Suppose $s_1 = s \upharpoonright$ (the carrier of S_1) and $s_2 = s \upharpoonright$ (the carrier of S_2). Then $\mathrm{Following}(s) = \mathrm{Following}(s_2) {+\cdot} \mathrm{Following}(s_1)$.

(41) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $\mathrm{InputVertices}(S_1) \subseteq \mathrm{InputVertices}(S_2)$ and $S = S_1 {+\cdot} S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1 {+\cdot} A_2$. Let s be a state of A, and let s_1 be a state of A_1, and let s_2 be a state of A_2. Suppose $s_1 = s \upharpoonright$ (the carrier of S_1) and $s_2 = s \upharpoonright$ (the carrier of S_2). Then $\mathrm{Following}(s) = \mathrm{Following}(s_2) {+\cdot} \mathrm{Following}(s_1)$.

(42) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $\mathrm{InputVertices}(S_2) \subseteq \mathrm{InputVertices}(S_1)$ and $S = S_1 {+\cdot} S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1 {+\cdot} A_2$. Let s be a state of A, and let s_1 be a state of A_1, and let s_2 be a state of A_2. Suppose $s_1 = s \upharpoonright$ (the carrier of S_1) and $s_2 = s \upharpoonright$ (the carrier of S_2). Then $\mathrm{Following}(s) = \mathrm{Following}(s_1) {+\cdot} \mathrm{Following}(s_2)$.

3. Signatures with One Operation

Let A, B be non empty sets and let a be an element of A. Then $B \longmapsto a$ is a function from B into A.

Let f be a set, let p be a finite sequence, and let x be a set. The functor $\mathrm{1GateCircStr}(p, f, x)$ yields a non void strict many sorted signature and is defined by the conditions (Def.5).

(Def.5) (i) The carrier of $\mathrm{1GateCircStr}(p, f, x) = \mathrm{rng}\, p \cup \{x\}$,

(ii) the operation symbols of $\mathrm{1GateCircStr}(p, f, x) = \{\langle p,\ f \rangle\}$,

(iii) (the arity of 1GateCircStr(p, f, x))$(\langle p,\ f\rangle) = p$, and

(iv) (the result sort of 1GateCircStr(p, f, x))$(\langle p,\ f\rangle) = x$.

Let f be a set, let p be a finite sequence, and let x be a set. Note that 1GateCircStr(p, f, x) is non empty.

The following propositions are true:

(43) Let f, x be sets and let p be a finite sequence. Then the arity of 1GateCircStr$(p, f, x) = \{\langle p,\ f\rangle\} \longmapsto p$ and the result sort of 1GateCircStr$(p, f, x) = \{\langle p,\ f\rangle\} \longmapsto x$.

(44) Let f, x be sets, and let p be a finite sequence, and let g be a gate of 1GateCircStr(p, f, x). Then $g = \langle p,\ f\rangle$ and Arity$(g) = p$ and the result sort of $g = x$.

(45) For all sets f, x and for every finite sequence p holds InputVertices (1GateCircStr(p, f, x)) $=$ rng $p \setminus \{x\}$ and InnerVertices (1GateCircStr(p, f, x)) $= \{x\}$.

Let f be a set and let p be a finite sequence. The functor 1GateCircStr(p, f) yielding a non void strict many sorted signature is defined by the conditions (Def.6).

(Def.6) (i) The carrier of 1GateCircStr$(p, f) = \mathrm{rng}\, p \cup \{\langle p,\ f\rangle\}$,

(ii) the operation symbols of 1GateCircStr$(p, f) = \{\langle p,\ f\rangle\}$,

(iii) (the arity of 1GateCircStr(p, f))$(\langle p,\ f\rangle) = p$, and

(iv) (the result sort of 1GateCircStr(p, f))$(\langle p,\ f\rangle) = \langle p,\ f\rangle$.

Let f be a set and let p be a finite sequence. Note that 1GateCircStr(p, f) is non empty.

One can prove the following propositions:

(46) For every set f and for every finite sequence p holds 1GateCircStr$(p, f) =$ 1GateCircStr$(p, f, \langle p,\ f\rangle)$.

(47) Let f be a set and let p be a finite sequence. Then the arity of 1GateCircStr$(p, f) = \{\langle p,\ f\rangle\} \longmapsto p$ and the result sort of 1GateCircStr$(p, f) = \{\langle p,\ f\rangle\} \longmapsto \langle p,\ f\rangle$.

(48) Let f be a set, and let p be a finite sequence, and let g be a gate of 1GateCircStr(p, f). Then $g = \langle p,\ f\rangle$ and Arity$(g) = p$ and the result sort of $g = g$.

(49) For every set f and for every finite sequence p holds InputVertices (1GateCircStr(p, f)) $= \mathrm{rng}\, p$ and InnerVertices(1GateCircStr(p, f)) $= \{\langle p,\ f\rangle\}$.

(50) For every set f and for every finite sequence p and for every set x such that $x \in \mathrm{rng}\, p$ holds $\mathrm{rk}(x) \in \mathrm{rk}(\langle p,\ f\rangle)$.

(51) For every set f and for all finite sequences p, q holds 1GateCircStr$(p, f) \approx$ 1GateCircStr(q, f).

4. Unsplit Condition

A many sorted signature is unsplit if:

(Def.7) The result sort of it $= \mathrm{id}_{(\text{the operation symbols of it})}$.

A many sorted signature has arity held in gates if:

(Def.8) For every set g such that $g \in$ the operation symbols of it holds $g = \langle$(the arity of it)(g), $g_{\mathbf{2}}\rangle$.

A many sorted signature has Boolean denotation held in gates if it satisfies the condition (Def.9).

(Def.9) Let g be a set. Suppose $g \in$ the operation symbols of it. Let p be a finite sequence. Suppose $p =$ (the arity of it)(g). Then there exists a function f from $\mathit{Boolean}^{\operatorname{len} p}$ into $\mathit{Boolean}$ such that $g = \langle g_{\mathbf{1}}, f\rangle$.

Let S be a non empty many sorted signature. An algebra over S has denotation held in gates if:

(Def.10) For every set g such that $g \in$ the operation symbols of S holds $g = \langle g_{\mathbf{1}}$, (the characteristics of it)$(g)\rangle$.

A non empty many sorted signature has denotation held in gates if:

(Def.11) There exists algebra over it which has denotation held in gates.

One can verify that every non empty many sorted signature which has Boolean denotation held in gates has also denotation held in gates.

The following two propositions are true:

(52) Let S be a non empty many sorted signature. Then S is unsplit if and only if for every set o such that $o \in$ the operation symbols of S holds (the result sort of S)$(o) = o$.

(53) Let S be a non empty many sorted signature. Suppose S is unsplit. Then the operation symbols of $S \subseteq$ the carrier of S.

Let us note that every non empty many sorted signature which is unsplit is also circuit-like.

The following proposition is true

(54) For every set f and for every finite sequence p holds $\mathrm{1GateCircStr}(p, f)$ is unsplit and has arity held in gates.

Let f be a set and let p be a finite sequence. Observe that $\mathrm{1GateCircStr}(p, f)$ is unsplit and has arity held in gates.

Let us observe that there exists a many sorted signature which is unsplit non void strict and non empty and has arity held in gates.

One can prove the following propositions:

(55) For all unsplit non empty many sorted signatures S_1, S_2 with arity held in gates holds $S_1 \approx S_2$.

(56) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be an algebra over S_1, and let A_2 be an algebra over S_2. Suppose A_1 has de-

notation held in gates and A_2 has denotation held in gates. Then the characteristics of $A_1 \approx$ the characteristics of A_2.

(57) For all unsplit non empty many sorted signatures S_1, S_2 holds $S_1 +\cdot S_2$ is unsplit.

Let S_1, S_2 be unsplit non empty many sorted signatures. Observe that $S_1 +\cdot S_2$ is unsplit.

We now state the proposition

(58) For all non empty many sorted signatures S_1, S_2 with arity held in gates holds $S_1 +\cdot S_2$ has arity held in gates.

Let S_1, S_2 be non empty many sorted signatures with arity held in gates. Note that $S_1 +\cdot S_2$ has arity held in gates.

The following proposition is true

(59) Let S_1, S_2 be non empty many sorted signatures. Suppose S_1 has Boolean denotation held in gates and S_2 has Boolean denotation held in gates. Then $S_1 +\cdot S_2$ has Boolean denotation held in gates.

5. One Gate Circuits

Let n be a natural number. A finite sequence is said to be a finite sequence with length n if:

(Def.12) $\operatorname{len} \mathrm{it} = n$.

Let n be a natural number, let X, Y be non empty sets, let f be a function from X^n into Y, let p be a finite sequence with length n, and let x be a set. Let us assume that if $x \in \operatorname{rng} p$, then $X = Y$. The functor $1\mathrm{GateCircuit}(p, f, x)$ yielding a strict non-empty algebra over $1\mathrm{GateCircStr}(p, f, x)$ is defined by:

(Def.13) The sorts of $1\mathrm{GateCircuit}(p, f, x) = (\operatorname{rng} p \longmapsto X) +\cdot (\{x\} \longmapsto Y)$ and (the characteristics of $1\mathrm{GateCircuit}(p, f, x))(\langle p,\ f\rangle) = f$.

Let n be a natural number, let X be a non empty set, let f be a function from X^n into X, and let p be a finite sequence with length n. The functor $1\mathrm{GateCircuit}(p, f)$ yielding a strict non-empty algebra over $1\mathrm{GateCircStr}(p, f)$ is defined as follows:

(Def.14) The sorts of $1\mathrm{GateCircuit}(p, f) =$ (the carrier of $1\mathrm{GateCircStr}(p, f)) \longmapsto (X)$ and (the characteristics of $1\mathrm{GateCircuit}(p, f))(\langle p,\ f\rangle) = f$.

Next we state the proposition

(60) Let n be a natural number, and let X be a non empty set, and let f be a function from X^n into X, and let p be a finite sequence with length n. Then $1\mathrm{GateCircuit}(p, f)$ has denotation held in gates and $1\mathrm{GateCircStr}(p, f)$ has denotation held in gates.

Let n be a natural number, let X be a non empty set, let f be a function from X^n into X, and let p be a finite sequence with length n. One can verify that 1GateCircuit(p, f) has denotation held in gates and 1GateCircStr(p, f) has denotation held in gates.

One can prove the following proposition

(61) Let n be a natural number, and let p be a finite sequence with length n, and let f be a function from $Boolean^n$ into $Boolean$. Then 1GateCircStr(p, f) has Boolean denotation held in gates.

Let n be a natural number, let f be a function from $Boolean^n$ into $Boolean$, and let p be a finite sequence with length n. Note that 1GateCircStr(p, f) has Boolean denotation held in gates.

One can check that there exists a many sorted signature which is non empty and has Boolean denotation held in gates.

Let S_1, S_2 be non empty many sorted signatures with Boolean denotation held in gates. Observe that $S_1 +\cdot S_2$ has Boolean denotation held in gates.

One can prove the following proposition

(62) Let n be a natural number, and let X be a non empty set, and let f be a function from X^n into X, and let p be a finite sequence with length n. Then the characteristics of 1GateCircuit$(p, f) = \{\langle p, f\rangle\} \longmapsto f$ and for every vertex v of 1GateCircStr(p, f) holds (the sorts of 1GateCircuit$(p, f))(v) = X$.

Let n be a natural number, let X be a non empty finite set, let f be a function from X^n into X, and let p be a finite sequence with length n. One can check that 1GateCircuit(p, f) is locally-finite.

Next we state two propositions:

(63) Let n be a natural number, and let X be a non empty set, and let f be a function from X^n into X, and let p, q be finite sequences with length n. Then 1GateCircuit$(p, f) \approx$ 1GateCircuit(q, f).

(64) Let n be a natural number, and let X be a finite non empty set, and let f be a function from X^n into X, and let p be a finite sequence with length n, and let s be a state of 1GateCircuit(p, f). Then (Following$(s))(\langle p, f\rangle) = f(s \cdot p)$.

Let X be a non empty set. Observe that there exists a non empty subset of X which is finite.

6. Boolean Circuits

Boolean is a finite non empty subset of $\mathbb{N}$.

Let S be a non empty many sorted signature. An algebra over S is Boolean if:

(Def.15) For every vertex v of S holds (the sorts of it)$(v) = Boolean$.

Next we state the proposition

(65) Let S be a non empty many sorted signature and let A be an algebra over S. Then A is Boolean if and only if the sorts of $A =$ (the carrier of $S) \longmapsto \mathit{Boolean}$.

Let S be a non empty many sorted signature. Note that every algebra over S which is Boolean is also non-empty and locally-finite.

One can prove the following three propositions:

(66) Let S be a non empty many sorted signature and let A be an algebra over S. Then A is Boolean if and only if rng (the sorts of A) $\subseteq \{\mathit{Boolean}\}$.

(67) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be an algebra over S_1, and let A_2 be an algebra over S_2. Suppose A_1 is Boolean and A_2 is Boolean. Then the sorts of $A_1 \approx$ the sorts of A_2.

(68) Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates, and let A_1 be an algebra over S_1, and let A_2 be an algebra over S_2. Suppose A_1 is Boolean and has denotation held in gates and A_2 is Boolean and has denotation held in gates. Then $A_1 \approx A_2$.

Let S be a non empty many sorted signature. One can check that there exists a strict algebra over S which is Boolean.

We now state three propositions:

(69) Let n be a natural number, and let f be a function from $\mathit{Boolean}^n$ into $\mathit{Boolean}$, and let p be a finite sequence with length n. Then $\mathrm{1GateCircuit}(p, f)$ is Boolean.

(70) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be a Boolean algebra over S_1, and let A_2 be a Boolean algebra over S_2. Then $A_1 +\cdot A_2$ is Boolean.

(71) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2. Suppose A_1 has denotation held in gates and A_2 has denotation held in gates and the sorts of $A_1 \approx$ the sorts of A_2. Then $A_1 +\cdot A_2$ has denotation held in gates.

Let us observe that there exists a non empty many sorted signature which is unsplit non void and strict and has arity held in gates, denotation held in gates, and Boolean denotation held in gates.

Let S be a non empty many sorted signature with Boolean denotation held in gates. Note that there exists a strict algebra over S which is Boolean and has denotation held in gates.

Let S_1, S_2 be unsplit non void non empty many sorted signatures with Boolean denotation held in gates, let A_1 be a Boolean circuit of S_1 with denotation held in gates, and let A_2 be a Boolean circuit of S_2 with denotation held in gates. One can verify that $A_1 +\cdot A_2$ is Boolean and has denotation held in gates.

Let n be a natural number, let X be a finite non empty set, let f be a function from X^n into X, and let p be a finite sequence with length n. Observe that there exists a circuit of $\mathrm{1GateCircStr}(p, f)$ which is strict and non-empty and has denotation held in gates.

Let n be a natural number, let X be a finite non empty set, let f be a function from X^n into X, and let p be a finite sequence with length n. Note that 1GateCircuit(p, f) has denotation held in gates.

One can prove the following proposition

(72) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates, and let A_1 be a Boolean circuit of S_1 with denotation held in gates, and let A_2 be a Boolean circuit of S_2 with denotation held in gates, and let s be a state of $A_1 +\cdot A_2$, and let v be a vertex of $S_1 +\cdot S_2$. Then

(i) for every state s_1 of A_1 such that $s_1 = s \restriction$ (the carrier of S_1) holds if $v \in$ InnerVertices(S_1) or $v \in$ the carrier of S_1 and $v \in$ InputVertices$(S_1 +\cdot S_2)$, then (Following(s))(v) = (Following(s_1))(v), and

(ii) for every state s_2 of A_2 such that $s_2 = s \restriction$ (the carrier of S_2) holds if $v \in$ InnerVertices(S_2) or $v \in$ the carrier of S_2 and $v \in$ InputVertices$(S_1 +\cdot S_2)$, then (Following(s))(v) = (Following(s_2))(v).

References

[1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[2] Grzegorz Bancerek. Tarski's classes and ranks. *Formalized Mathematics*, 1(**3**):563–567, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[10] Anna Lango and Grzegorz Bancerek. Product of families of groups and vector spaces. *Formalized Mathematics*, 3(**2**):235–240, 1992.
[11] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.
[12] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Introduction to circuits, II. *Formalized Mathematics*, 5(**2**):273–278, 1996.
[13] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Introduction to circuits, I. *Formalized Mathematics*, 5(**2**):227–232, 1996.
[14] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, II. *Formalized Mathematics*, 5(**2**):215–220, 1996.
[15] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Formalized Mathematics*, 5(**2**):167–172, 1996.
[16] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[17] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[18] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.
[19] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.

[20] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[21] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[22] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.

[23] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[24] Edmund Woronowicz. Many–argument relations. *Formalized Mathematics*, 1(**4**):733–737, 1990.

[25] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received May 11, 1995