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Introduction to Circuits, I [1]

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Summary. This article is the third in a series of four articles (preceded by [19,20] and continued in [18]) about modelling circuits by many sorted algebras.

A circuit is defined as a locally-finite algebra over a circuit-like many sorted signature. For circuits we define notions of input function and of circuit state which are later used (see [18]) to define circuit computations. For circuits over monotonic signatures we introduce notions of vertex size and vertex depth that characterize certain graph properties of circuit's signature in terms of elements of its free envelope algebra. The depth of a finite circuit is defined as the maximal depth over its vertices.

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The terminology and notation used in this paper are introduced in the following papers: [24], [27], [3], [16], [28], [12], [9], [29], [15], [25], [1], [7], [26], [13], [2], [4], [6], [8], [5], [14], [10], [23], [22], [11], [17], [21], [19], and [20].

1. Circuit State

Let S be a non void circuit-like non empty many sorted signature. A circuit of S is a locally-finite algebra over S.

In the sequel I_1 will denote a circuit-like non void non empty many sorted signature.

Let us consider I_1 and let S_1 be a non-empty circuit of I_1. The functor Set-Constants(S_1) yielding a many sorted set indexed by SortsWithConstants(I_1) is defined as follows:

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(Def.1) For every vertex x of I_1 such that $x \in \mathrm{dom}\,\text{Set-Constants}(S_1)$ holds $(\text{Set-Constants}(S_1))(x) \in \text{Constants}(S_1, x)$.

The following proposition is true

(1) Given I_1, and let S_1 be a non-empty circuit of I_1, and let v be a vertex of I_1, and let e be an element of (the sorts of S_1)(v). If $v \in \text{SortsWithConstants}(I_1)$ and $e \in \text{Constants}(S_1, v)$, then $(\text{Set-Constants}(S_1))(v) = e$.

Let us consider I_1 and let C_1 be a circuit of I_1. An input function of C_1 is a many sorted function from $\text{InputVertices}(I_1) \longmapsto \mathbb{N}$ into (the sorts of C_1) $\restriction$ $\text{InputVertices}(I_1)$.

The following proposition is true

(2) Given I_1, and let S_1 be a non-empty circuit of I_1, and let I_2 be an input function of S_1, and let n be a natural number. If I_1 has input vertices, then $(\text{commute}(I_2))(n)$ is an input assignment of S_1.

Let us consider I_1. Let us assume that I_1 has input vertices. Let S_1 be a non-empty circuit of I_1, let I_2 be an input function of S_1, and let n be a natural number. The functor n-*th*-input(I_2) yields an input assignment of S_1 and is defined by:

(Def.2) n-*th*-input$(I_2) = (\text{commute}(I_2))(n)$.

The following proposition is true

(3) Given I_1, and let S_1 be a non-empty circuit of I_1, and let I_2 be an input function of S_1, and let n be a natural number. If I_1 has input vertices, then n-*th*-input$(I_2) = (\text{commute}(I_2))(n)$.

Let us consider I_1 and let S_1 be a circuit of I_1. A state of S_1 is an element of $\prod$(the sorts of S_1).

The following propositions are true:

(4) For every I_1 and for every non-empty circuit S_1 of I_1 and for every state s of S_1 holds $\mathrm{dom}\, s =$ the carrier of I_1.

(5) Given I_1, and let S_1 be a non-empty circuit of I_1, and let s be a state of S_1, and let v be a vertex of I_1. Then $s(v) \in$ (the sorts of S_1)(v).

Let us consider I_1, let S_1 be a non-empty circuit of I_1, let s be a state of S_1, and let o be an operation symbol of I_1. The functor o depends-on-in s yields an element of $\text{Args}(o, S_1)$ and is defined as follows:

(Def.3) o depends-on-in $s = s \cdot \text{Arity}(o)$.

In the sequel I_1 will be a monotonic circuit-like non void non empty many sorted signature.

The following proposition is true

(6) Given I_1, and let S_1 be a locally-finite non-empty algebra over I_1, and let v, w be vertices of I_1, and let e_1 be an element of (the sorts of $\text{FreeEnvelope}(S_1)$)(v), and let q_1 be a decorated tree yielding finite sequence. Suppose $v \in \text{InnerVertices}(I_1)$ and $e_1 = \langle$the action at v, the carrier of $I_1\rangle$-tree(q_1). Let k be a natural number. If $k \in \mathrm{dom}\, q_1$ and

$q_1(k) \in$ (the sorts of FreeEnvelope(S_1))(w), then $w = \pi_k$ Arity(the action at v).

Let us consider I_1, let S_1 be a locally-finite non-empty algebra over I_1, and let v be a vertex of I_1. Note that every element of the sorts of FreeEnvelope$(S_1)(v)$ is finite non empty function-like and relation-like.

Let us consider I_1, let S_1 be a locally-finite non-empty algebra over I_1, and let v be a vertex of I_1. Observe that every element of the sorts of FreeEnvelope$(S_1)(v)$ is decorated tree-like.

Next we state four propositions:

(7) Given I_1, and let S_1 be a locally-finite non-empty algebra over I_1, and let v, w be vertices of I_1, and let e_1 be an element of (the sorts of FreeEnvelope(S_1))(v), and let e_2 be an element of (the sorts of FreeEnvelope(S_1))(w), and let q_1 be a decorated tree yielding finite sequence, and let k_1 be a natural number. Suppose $v \in$ InnerVertices$(I_1) \setminus$ SortsWithConstants(I_1) and $e_1 = \langle$the action at v, the carrier of $I_1\rangle$-tree(q_1) and $k_1 + 1 \in \operatorname{dom} q_1$ and $q_1(k_1 + 1) \in$ (the sorts of FreeEnvelope(S_1))(w). Then $e_1(\langle k_1\rangle/e_2) \in$ (the sorts of FreeEnvelope(S_1))(v).

(8) Given I_1, and let A be a locally-finite non-empty algebra over I_1, and let v be an element of the carrier of I_1, and let e be an element of (the sorts of FreeEnvelope(A))(v). Suppose $1 < \operatorname{card} e$. Then there exists an operation symbol o of I_1 such that $e(\varepsilon) = \langle o$, the carrier of $I_1\rangle$.

(9) Let I_1 be a non void circuit-like non empty many sorted signature, and let S_1 be a non-empty circuit of I_1, and let s be a state of S_1, and let o be an operation symbol of I_1. Then $(\mathrm{Den}(o, S_1))(o\,\text{depends-on-in}\,s) \in$ (the sorts of S_1)(the result sort of o).

(10) Given I_1, and let A be a non-empty circuit of I_1, and let v be a vertex of I_1, and let e be an element of (the sorts of FreeEnvelope(A))(v). Suppose $e(\varepsilon) = \langle$the action at v, the carrier of $I_1\rangle$. Then there exists a decorated tree yielding finite sequence p such that $e = \langle$the action at v, the carrier of $I_1\rangle$-tree(p).

2. Vertex Size

Let I_1 be a monotonic non void non empty many sorted signature, let A be a locally-finite non-empty algebra over I_1, and let v be a sort symbol of I_1. One can verify that (the sorts of FreeEnvelope(A))(v) is finite.

Let us consider I_1, let A be a locally-finite non-empty algebra over I_1, and let v be a sort symbol of I_1. The functor size(v, A) yielding a natural number is defined as follows:

(Def.4) There exists a finite non empty subset s of $\mathbb{N}$ such that $s = \{\operatorname{card} t : t$ ranges over elements of (the sorts of FreeEnvelope(A))$(v)\}$ and size$(v, A) = \max s$.

Next we state four propositions:

(11) Given I_1, and let A be a locally-finite non-empty algebra over I_1, and let v be an element of the carrier of I_1. Then $\mathrm{size}(v, A) = 1$ if and only if $v \in \mathrm{InputVertices}(I_1) \cup \mathrm{SortsWithConstants}(I_1)$.

(12) Given I_1, and let S_1 be a locally-finite non-empty algebra over I_1, and let v, w be vertices of I_1, and let e_1 be an element of (the sorts of FreeEnvelope(S_1))(v), and let e_2 be an element of (the sorts of FreeEnvelope(S_1))(w), and let q_1 be a decorated tree yielding finite sequence. Suppose $v \in \mathrm{InnerVertices}(I_1) \setminus \mathrm{SortsWithConstants}(I_1)$ and $\mathrm{card}\, e_1 = \mathrm{size}(v, S_1)$ and $e_1 = \langle$the action at v, the carrier of $I_1\rangle$-tree(q_1) and $e_2 \in \mathrm{rng}\, q_1$. Then $\mathrm{card}\, e_2 = \mathrm{size}(w, S_1)$.

(13) Given I_1, and let A be a locally-finite non-empty algebra over I_1, and let v be a vertex of I_1, and let e be an element of (the sorts of FreeEnvelope(A))(v). Suppose $v \in \mathrm{InnerVertices}(I_1) \setminus \mathrm{SortsWithConstants}(I_1)$ and $\mathrm{card}\, e = \mathrm{size}(v, A)$. Then there exists a decorated tree yielding finite sequence q such that $e = \langle$the action at v, the carrier of $I_1\rangle$-tree(q).

(14) Given I_1, and let A be a locally-finite non-empty algebra over I_1, and let v be a vertex of I_1, and let e be an element of (the sorts of FreeEnvelope(A))(v). Suppose $v \in \mathrm{InnerVertices}(I_1) \setminus \mathrm{SortsWithConstants}(I_1)$ and $\mathrm{card}\, e = \mathrm{size}(v, A)$. Then there exists an operation symbol o of I_1 such that $e(\varepsilon) = \langle o$, the carrier of $I_1\rangle$.

Let S be a non void non empty many sorted signature, let A be a locally-finite non-empty algebra over S, let v be a sort symbol of S, and let e be an element of (the sorts of FreeEnvelope(A))(v). The functor depth(e) yielding a natural number is defined as follows:

(Def.5) There exists an element e' of (the sorts of Free(the sorts of A))(v) such that $e = e'$ and $\mathrm{depth}(e) = \mathrm{depth}(e')$.

The following propositions are true:

(15) Given I_1, and let A be a locally-finite non-empty algebra over I_1, and let v, w be elements of the carrier of I_1. If $v \in \mathrm{InnerVertices}(I_1)$ and $w \in \mathrm{rng}\,\mathrm{Arity}$(the action at v), then $\mathrm{size}(w, A) < \mathrm{size}(v, A)$.

(16) For every I_1 and for every locally-finite non-empty algebra A over I_1 and for every sort symbol v of I_1 holds $\mathrm{size}(v, A) > 0$.

(17) Given I_1, and let A be a non-empty circuit of I_1, and let v be a vertex of I_1, and let e be an element of (the sorts of FreeEnvelope(A))(v), and let p be a decorated tree yielding finite sequence. Suppose that

(i) $v \in \mathrm{InnerVertices}(I_1)$,

(ii) $e = \langle$the action at v, the carrier of $I_1\rangle$-tree(p), and

(iii) for every natural number k such that $k \in \mathrm{dom}\, p$ there exists an element e_3 of (the sorts of FreeEnvelope(A))($\pi_k\,\mathrm{Arity}$(the action at v)) such that $e_3 = p(k)$ and $\mathrm{card}\, e_3 = \mathrm{size}(\pi_k\,\mathrm{Arity}(\text{the action at } v), A)$.

Then $\mathrm{card}\, e = \mathrm{size}(v, A)$.

3. Vertex and Circuit Depth

Let S be a monotonic non void non empty many sorted signature, let A be a locally-finite non-empty algebra over S, and let v be a sort symbol of S. The functor $\mathrm{depth}(v, A)$ yields a natural number and is defined by:

(Def.6) There exists a finite non empty subset s of $\mathbb{N}$ such that $s = \{\mathrm{depth}(t) : t$ ranges over elements of (the sorts of $\mathrm{FreeEnvelope}(A))(v)\}$ and $\mathrm{depth}(v, A) = \max s$.

Let I_1 be a finite monotonic circuit-like non void non empty many sorted signature and let A be a non-empty circuit of I_1. The functor $\mathrm{depth}(A)$ yielding a natural number is defined by the condition (Def.7).

(Def.7) There exists a finite non empty subset D_1 of $\mathbb{N}$ such that $D_1 = \{\mathrm{depth}(v, A) : v$ ranges over elements of the carrier of I_1, $v \in$ the carrier of $I_1\}$ and $\mathrm{depth}(A) = \max D_1$.

The following three propositions are true:

(18) Let I_1 be a finite monotonic circuit-like non void non empty many sorted signature, and let A be a non-empty circuit of I_1, and let v be a vertex of I_1. Then $\mathrm{depth}(v, A) \leq \mathrm{depth}(A)$.

(19) Given I_1, and let A be a non-empty circuit of I_1, and let v be a vertex of I_1. Then $\mathrm{depth}(v, A) = 0$ if and only if $v \in \mathrm{InputVertices}(I_1)$ or $v \in \mathrm{SortsWithConstants}(I_1)$.

(20) Given I_1, and let A be a locally-finite non-empty algebra over I_1, and let v, v_1 be sort symbols of I_1. If $v \in \mathrm{InnerVertices}(I_1)$ and $v_1 \in \mathrm{rng}\,\mathrm{Arity}($the action at $v)$, then $\mathrm{depth}(v_1, A) < \mathrm{depth}(v, A)$.

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