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Ideals

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Summary. The dual concept to filters (see [2,3]) i.e. ideals of a lattice is introduced.

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The articles [12], [14], [13], [4], [15], [6], [10], [9], [7], [5], [16], [8], [2], [11], [3], and [1] provide the notation and terminology for this paper.

1. Some Properties of the Restriction of Binary Operations

In this paper D is a non empty set.

We now state several propositions:

(1) Let D be a non empty set, and let S be a non empty subset of D, and let f be a binary operation on D, and let g be a binary operation on S. Suppose $g = f \upharpoonright [\!: S, S :\!]$. Then

(i) if f is commutative, then g is commutative,

(ii) if f is idempotent, then g is idempotent, and

(iii) if f is associative, then g is associative.

(2) Let D be a non empty set, and let S be a non empty subset of D, and let f be a binary operation on D, and let g be a binary operation on S, and let d be an element of D, and let d' be an element of S. Suppose $g = f \upharpoonright [\!: S, S :\!]$ and $d' = d$. Then

(i) if d is a left unity w.r.t. f, then d' is a left unity w.r.t. g,

(ii) if d is a right unity w.r.t. f, then d' is a right unity w.r.t. g, and

(iii) if d is a unity w.r.t. f, then d' is a unity w.r.t. g.

(3) Let D be a non empty set, and let S be a non empty subset of D, and let f_1, f_2 be binary operations on D, and let g_1, g_2 be binary operations on S. Suppose $g_1 = f_1 \upharpoonright [\!: S, S :\!]$ and $g_2 = f_2 \upharpoonright [\!: S, S :\!]$. Then

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(i) if f_1 is left distributive w.r.t. f_2, then g_1 is left distributive w.r.t. g_2, and

(ii) if f_1 is right distributive w.r.t. f_2, then g_1 is right distributive w.r.t. g_2.

(4) Let D be a non empty set, and let S be a non empty subset of D, and let f_1, f_2 be binary operations on D, and let g_1, g_2 be binary operations on S. Suppose $g_1 = f_1 \restriction [\!: S, S :\!]$ and $g_2 = f_2 \restriction [\!: S, S :\!]$. If f_1 is distributive w.r.t. f_2, then g_1 is distributive w.r.t. g_2.

(5) Let D be a non empty set, and let S be a non empty subset of D, and let f_1, f_2 be binary operations on D, and let g_1, g_2 be binary operations on S. If $g_1 = f_1 \restriction [\!: S, S :\!]$ and $g_2 = f_2 \restriction [\!: S, S :\!]$, then if f_1 absorbs f_2, then g_1 absorbs g_2.

2. Closed Subsets of a Lattice

Let D be a non empty set and let X_1, X_2 be subsets of D. Let us observe that $X_1 = X_2$ if and only if:

(Def.1) For every element x of D holds $x \in X_1$ iff $x \in X_2$.

For simplicity we follow the rules: L will denote a lattice, p, q, r will denote elements of the carrier of L, p', q' will denote elements of the carrier of L°, and x will be arbitrary.

Next we state several propositions:

(6) Let L_1, L_2 be lattice structures. Suppose the lattice structure of $L_1 =$ the lattice structure of L_2. Then ${L_1}^\circ = {L_2}^\circ$.

(7) $(L^\circ)^\circ =$ the lattice structure of L.

(8) Let L_1, L_2 be non empty lattice structures. Suppose the lattice structure of $L_1 =$ the lattice structure of L_2. Let a_1, b_1 be elements of the carrier of L_1 and let a_2, b_2 be elements of the carrier of L_2. Suppose $a_1 = a_2$ and $b_1 = b_2$. Then $a_1 \sqcup b_1 = a_2 \sqcup b_2$ and $a_1 \sqcap b_1 = a_2 \sqcap b_2$ and $a_1 \sqsubseteq b_1$ iff $a_2 \sqsubseteq b_2$.

(9) Let L_1, L_2 be lower bound lattices. Suppose the lattice structure of $L_1 =$ the lattice structure of L_2. Then $\bot_{(L_1)} = \bot_{(L_2)}$.

(10) Let L_1, L_2 be upper bound lattices. Suppose the lattice structure of $L_1 =$ the lattice structure of L_2. Then $\top_{(L_1)} = \top_{(L_2)}$.

(11) Let L_1, L_2 be complemented lattices. Suppose the lattice structure of $L_1 =$ the lattice structure of L_2. Let a_1, b_1 be elements of the carrier of L_1 and let a_2, b_2 be elements of the carrier of L_2. If $a_1 = a_2$ and $b_1 = b_2$ and a_1 is a complement of b_1, then a_2 is a complement of b_2.

(12) Let L_1, L_2 be Boolean lattices. Suppose the lattice structure of $L_1 =$ the lattice structure of L_2. Let a be an element of the carrier of L_1 and let b be an element of the carrier of L_2. If $a = b$, then $a^{\rm c} = b^{\rm c}$.

Let us consider L. A subset of the carrier of L is said to be a closed subset of L if:

(Def.2) For all p, q such that $p \in$ it and $q \in$ it holds $p \sqcap q \in$ it and $p \sqcup q \in$ it.

Let us consider L. Observe that there exists a closed subset of L which is non empty.

The following two propositions are true:

(13) Let X be a subset of the carrier of L. Suppose that for all p, q holds $p \in X$ and $q \in X$ iff $p \sqcap q \in X$. Then X is a closed subset of L.

(14) Let X be a subset of the carrier of L. Suppose that for all p, q holds $p \in X$ and $q \in X$ iff $p \sqcup q \in X$. Then X is a closed subset of L.

Let us consider L. Then $[L)$ is a filter of L. Let p be an element of the carrier of L. Then $[p)$ is a filter of L.

Let us consider L and let D be a non empty subset of the carrier of L. Then $[D)$ is a filter of L.

Let L be a distributive lattice and let F_1, F_2 be filters of L. Then $F_1 \sqcap F_2$ is a filter of L.

Let us consider L. A non empty closed subset of L is called an ideal of L if:

(Def.3) $p \in$ it and $q \in$ it iff $p \sqcup q \in$ it.

Next we state three propositions:

(15) Let X be a non empty subset of the carrier of L. Suppose that for all p, q holds $p \in X$ and $q \in X$ iff $p \sqcup q \in X$. Then X is an ideal of L.

(16) Let L_1, L_2 be lattices. Suppose the lattice structure of $L_1 =$ the lattice structure of L_2. Given x. If x is a filter of L_1, then x is a filter of L_2.

(17) Let L_1, L_2 be lattices. Suppose the lattice structure of $L_1 =$ the lattice structure of L_2. Given x. If x is an ideal of L_1, then x is an ideal of L_2.

Let us consider L, p. The functor p° yielding an element of the carrier of L° is defined by:

(Def.4) $p^\circ = p$.

Let us consider L and let p be an element of the carrier of L°. The functor ${}^\circ p$ yields an element of the carrier of L and is defined as follows:

(Def.5) ${}^\circ p = p$.

Next we state four propositions:

(18) ${}^\circ p^\circ = p$ and $({}^\circ p')^\circ = p'$.

(19) $p \sqcap q = p^\circ \sqcup q^\circ$ and $p \sqcup q = p^\circ \sqcap q^\circ$ and $p' \sqcap q' = {}^\circ p' \sqcup {}^\circ q'$ and $p' \sqcup q' = {}^\circ p' \sqcap {}^\circ q'$.

(20) $p \sqsubseteq q$ iff $q^\circ \sqsubseteq p^\circ$ and $p' \sqsubseteq q'$ iff ${}^\circ q' \sqsubseteq {}^\circ p'$.

(21) x is an ideal of L iff x is a filter of L°.

Let us consider L and let X be a subset of the carrier of L. The functor X° yielding a subset of the carrier of L° is defined as follows:

(Def.6) $X^\circ = X$.

Let us consider L and let X be a subset of the carrier of L°. The functor ${}^\circ X$ yielding a subset of the carrier of L is defined by:

(Def.7) $^\circ X = X$.

Let us consider L and let D be a non empty subset of the carrier of L. Observe that D° is non empty.

Let us consider L and let D be a non empty subset of the carrier of L°. Observe that $^\circ D$ is non empty.

Let us consider L and let S be a closed subset of L. Then S° is a closed subset of L°.

Let us consider L and let S be a non empty closed subset of L. Then S° is a non empty closed subset of L°.

Let us consider L and let S be a closed subset of L°. Then $^\circ S$ is a closed subset of L.

Let us consider L and let S be a non empty closed subset of L°. Then $^\circ S$ is a non empty closed subset of L.

Let us consider L and let F be a filter of L. Then F° is an ideal of L°.

Let us consider L and let F be a filter of L°. Then $^\circ F$ is an ideal of L.

Let us consider L and let I be an ideal of L. Then I° is a filter of L°.

Let us consider L and let I be an ideal of L°. Then $^\circ I$ is a filter of L.

We now state the proposition

(22) Let D be a non empty subset of the carrier of L. Then D is an ideal of L if and only if the following conditions are satisfied:
 (i) for all p, q such that $p \in D$ and $q \in D$ holds $p \sqcup q \in D$, and
 (ii) for all p, q such that $p \in D$ and $q \sqsubseteq p$ holds $q \in D$.

In the sequel I, J will be ideals of L and F will be a filter of L.

One can prove the following propositions:

(23) If $p \in I$, then $p \sqcap q \in I$ and $q \sqcap p \in I$.

(24) There exists p such that $p \in I$.

(25) If L is lower-bounded, then $\bot_L \in I$.

(26) If L is lower-bounded, then $\{\bot_L\}$ is an ideal of L.

(27) If $\{p\}$ is an ideal of L, then L is lower-bounded.

3. Ideals Generated by Subsets of a Lattice

Next we state the proposition

(28) The carrier of L is an ideal of L.

Let us consider L. The functor $(L]$ yielding an ideal of L is defined as follows:

(Def.8) $(L] =$ the carrier of L.

Let us consider L, p. The functor $(p]$ yields an ideal of L and is defined as follows:

(Def.9) $(p] = \{q : q \sqsubseteq p\}$.

We now state four propositions:

(29) $q \in (p]$ iff $q \sqsubseteq p$.

(30) $(p] = [p^\circ)$ and $(p^\circ] = [p)$.

(31) $p \in (p]$ and $p \sqcap q \in (p]$ and $q \sqcap p \in (p]$.

(32) If L is upper-bounded, then $(L] = (\top_L]$.

Let us consider L, I. We say that I is maximal if and only if:

(Def.10) $I \neq$ the carrier of L and for every J such that $I \subseteq J$ and $J \neq$ the carrier of L holds $I = J$.

One can prove the following four propositions:

(33) I is maximal iff I° is an ultrafilter.

(34) If L is upper-bounded, then for every I such that $I \neq$ the carrier of L there exists J such that $I \subseteq J$ and J is maximal.

(35) If there exists r such that $p \sqcup r \neq p$, then $(p] \neq$ the carrier of L.

(36) If L is upper-bounded and $p \neq \top_L$, then there exists I such that $p \in I$ and I is maximal.

In the sequel D denotes a non empty subset of the carrier of L and D' denotes a non empty subset of the carrier of L°.

Let us consider L, D. The functor $(D]$ yields an ideal of L and is defined as follows:

(Def.11) $D \subseteq (D]$ and for every I such that $D \subseteq I$ holds $(D] \subseteq I$.

We now state two propositions:

(37) $[D^\circ) = (D]$ and $[D) = (D^\circ]$ and $[{}^\circ D') = (D']$ and $[D') = ({}^\circ D']$.

(38) $(I] = I$.

In the sequel D_1, D_2 are non empty subsets of the carrier of L and D'_1, D'_2 are non empty subsets of the carrier of L°.

The following propositions are true:

(39) If $D_1 \subseteq D_2$, then $(D_1] \subseteq (D_2]$ and $((D]] \subseteq (D]$.

(40) If $p \in D$, then $(p] \subseteq (D]$.

(41) If $D = \{p\}$, then $(D] = (p]$.

(42) If L is upper-bounded and $\top_L \in D$, then $(D] = (L]$ and $(D] =$ the carrier of L.

(43) If L is upper-bounded and $\top_L \in I$, then $I = (L]$ and $I =$ the carrier of L.

Let us consider L, I. We say that I is prime if and only if:

(Def.12) $p \sqcap q \in I$ iff $p \in I$ or $q \in I$.

The following proposition is true

(44) I is prime iff I° is prime.

Let us consider L, D_1, D_2. The functor $D_1 \sqcup D_2$ yielding a non empty subset of the carrier of L is defined by:

(Def.13) $D_1 \sqcup D_2 = \{p \sqcup q : p \in D_1 \ \wedge\ q \in D_2\}$.

We now state four propositions:

(45) $D_1 \sqcup D_2 = {D_1}^\circ \sqcap {D_2}^\circ$ and ${D_1}^\circ \sqcup {D_2}^\circ = D_1 \sqcap D_2$ and $D'_1 \sqcup D'_2 = {}^\circ D'_1 \sqcap {}^\circ D'_2$ and ${}^\circ D'_1 \sqcup {}^\circ D'_2 = D'_1 \sqcap D'_2$.

(46) If $p \in D_1$ and $q \in D_2$, then $p \sqcup q \in D_1 \sqcup D_2$ and $q \sqcup p \in D_1 \sqcup D_2$.

(47) If $x \in D_1 \sqcup D_2$, then there exist p, q such that $x = p \sqcup q$ and $p \in D_1$ and $q \in D_2$.

(48) $D_1 \sqcup D_2 = D_2 \sqcup D_1$.

Let L be a distributive lattice and let I_1, I_2 be ideals of L. Then $I_1 \sqcup I_2$ is an ideal of L.

The following four propositions are true:

(49) $(D_1 \cup D_2] = ((D_1] \cup D_2]$ and $(D_1 \cup D_2] = (D_1 \cup (D_2]]$.

(50) $(I \cup J] = \{r : \bigvee_{p,q} \ r \sqsubseteq p \sqcup q \ \wedge \ p \in I \ \wedge \ q \in J\}$.

(51) $I \subseteq I \sqcup J$ and $J \subseteq I \sqcup J$.

(52) $(I \cup J] = (I \sqcup J]$.

We follow the rules: B denotes a Boolean lattice, I_3, J_1 denote ideals of B, and a, b denote elements of the carrier of B.

The following propositions are true:

(53) L is a complemented lattice iff L° is a complemented lattice.

(54) L is a Boolean lattice iff L° is a Boolean lattice.

Let B be a Boolean lattice. One can verify that B° is Boolean and lattice-like.

In the sequel a' will denote an element of the carrier of (B **qua** lattice)$^\circ$.

The following propositions are true:

(55) $(a^\circ)^{\rm c} = a^{\rm c}$ and $({}^\circ a')^{\rm c} = a'^{\rm c}$.

(56) $(I_3 \cup J_1] = I_3 \sqcup J_1$.

(57) I_3 is maximal iff $I_3 \neq$ the carrier of B and for every a holds $a \in I_3$ or $a^{\rm c} \in I_3$.

(58) $I_3 \neq (B]$ and I_3 is prime iff I_3 is maximal.

(59) If I_3 is maximal, then for every a holds $a \in I_3$ iff $a^{\rm c} \notin I_3$.

(60) If $a \neq b$, then there exists I_3 such that I_3 is maximal but $a \in I_3$ and $b \notin I_3$ or $a \notin I_3$ and $b \in I_3$.

In the sequel P denotes a non empty closed subset of L and o_1, o_2 denote binary operations on P.

One can prove the following two propositions:

(61) (i) (The join operation of L) $\restriction [\!: P, P :\!]$ is a binary operation on P, and

(ii) (the meet operation of L) $\restriction [\!: P, P :\!]$ is a binary operation on P.

(62) Suppose $o_1 =$ (the join operation of L) $\restriction [\!: P, P :\!]$ and $o_2 =$ (the meet operation of L) $\restriction [\!: P, P :\!]$. Then o_1 is commutative and associative and o_2 is commutative and associative and o_1 absorbs o_2 and o_2 absorbs o_1.

Let us consider L, p, q. Let us assume that $p \sqsubseteq q$. The functor $[p, q]$ yielding a non empty closed subset of L is defined by:

(Def.14) $[p, q] = \{r : p \sqsubseteq r \ \wedge \ r \sqsubseteq q\}$.

We now state several propositions:

(63) If $p \sqsubseteq q$, then $r \in [p, q]$ iff $p \sqsubseteq r$ and $r \sqsubseteq q$.

(64) If $p \sqsubseteq q$, then $p \in [p, q]$ and $q \in [p, q]$.

(65) $[p, p] = \{p\}$.

(66) If L is upper-bounded, then $[p) = [p, \top_L]$.

(67) If L is lower-bounded, then $(p] = [\bot_L, p]$.

(68) Let L_1, L_2 be lattices, and let F_1 be a filter of L_1, and let F_2 be a filter of L_2. Suppose the lattice structure of L_1 = the lattice structure of L_2 and $F_1 = F_2$. Then $\mathbb{L}_{(F_1)} = \mathbb{L}_{(F_2)}$.

4. Sublattices

Let us consider L. Let us note that the sublattice of L can be characterized by the following (equivalent) condition:

(Def.15) There exist P, o_1, o_2 such that

(i) o_1 = (the join operation of L) $\restriction [\!: P, P :\!]$,

(ii) o_2 = (the meet operation of L) $\restriction [\!: P, P :\!]$, and

(iii) the lattice structure of it $= \langle P, o_1, o_2 \rangle$.

The following proposition is true

(69) For every sublattice K of L holds every element of the carrier of K is an element of the carrier of L.

Let us consider L, P. The functor $\mathbb{L}_P^L$ yields a strict sublattice of L and is defined as follows:

(Def.16) There exist o_1, o_2 such that o_1 = (the join operation of L) $\restriction [\!: P, P :\!]$ and o_2 = (the meet operation of L) $\restriction [\!: P, P :\!]$ and $\mathbb{L}_P^L = \langle P, o_1, o_2 \rangle$.

Let us consider L and let l be a sublattice of L. Then l° is a strict sublattice of L°.

Next we state a number of propositions:

(70) $\mathbb{L}_F = \mathbb{L}_F^L$.

(71) $\mathbb{L}_P^L = (\mathbb{L}_{P^\circ}^{L^\circ})^\circ$.

(72) $\mathbb{L}_{(L]}^L$ = the lattice structure of L and $\mathbb{L}_{[L)}^L$ = the lattice structure of L.

(73) (i) The carrier of $\mathbb{L}_P^L = P$,

(ii) the join operation of $\mathbb{L}_P^L$ = (the join operation of L) $\restriction [\!: P, P :\!]$, and

(iii) the meet operation of $\mathbb{L}_P^L$ = (the meet operation of L) $\restriction [\!: P, P :\!]$.

(74) For all p, q and for all elements p', q' of the carrier of $\mathbb{L}_P^L$ such that $p = p'$ and $q = q'$ holds $p \sqcup q = p' \sqcup q'$ and $p \sqcap q = p' \sqcap q'$.

(75) For all p, q and for all elements p', q' of the carrier of $\mathbb{L}_P^L$ such that $p = p'$ and $q = q'$ holds $p \sqsubseteq q$ iff $p' \sqsubseteq q'$.

(76) If L is lower-bounded, then $\mathbb{L}_I^L$ is lower-bounded.

(77) If L is modular, then $\mathbb{L}_P^L$ is modular.

(78) If L is distributive, then $\mathbb{L}_P^L$ is distributive.

(79) If L is implicative and $p \sqsubseteq q$, then $\mathbb{L}^L_{[p,q]}$ is implicative.

(80) $\mathbb{L}^L_{(p]}$ is upper-bounded.

(81) $\top_{\mathbb{L}^L_{(p]}} = p$.

(82) If L is lower-bounded, then $\mathbb{L}^L_{(p]}$ is lower-bounded and $\perp_{\mathbb{L}^L_{(p]}} = \perp_L$.

(83) If L is lower-bounded, then $\mathbb{L}^L_{(p]}$ is bounded.

(84) If $p \sqsubseteq q$, then $\mathbb{L}^L_{[p,q]}$ is bounded and $\top_{\mathbb{L}^L_{[p,q]}} = q$ and $\perp_{\mathbb{L}^L_{[p,q]}} = p$.

(85) If L is a complemented lattice and modular, then $\mathbb{L}^L_{(p]}$ is a complemented lattice.

(86) If L is a complemented lattice and modular and $p \sqsubseteq q$, then $\mathbb{L}^L_{[p,q]}$ is a complemented lattice.

(87) If L is a Boolean lattice and $p \sqsubseteq q$, then $\mathbb{L}^L_{[p,q]}$ is a Boolean lattice.

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