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A Scheme for Extensions of Homomorphisms of Many Sorted Algebras

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Summary. The aim of this workis to provide a bridge between the theory of context-free grammars developed in [11], [6] and universally free manysorted algebras([17]. The third scheme proved in the article allows to prove that two homomorphisms equal on the set of free generators are equal. The first scheme is a slight modification of the scheme in [6] and the second is rather technical, but since it was useful for me, perhaps it might be useful for somebody else. The concept of flattening of a many sorted function F between two manysorted sets A and B (with common set of indices I) is introduced for A with mutually disjoint components (pairwise disjoint function – the concept introduced in [16]). This is a function on the union of A, that is equal to F on every component of A. A trivial many sorted algebra over a signature S is defined with sorts being singletons of corresponding sort symbols. It has mutually disjoint sorts.

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The notation and terminology used in this paper are introduced in the following articles: [20], [23], [24], [8], [9], [21], [5], [7], [14], [16], [3], [22], [2], [4], [1], [15], [11], [6], [10], [19], [13], [18], [17], and [12].

One can prove the following proposition

(1) For all functions f, g such that $g \in \prod f$ holds $\operatorname{rng} g \subseteq \bigcup f$.

The scheme *DTConstrUniq* concerns a non empty tree construction structure $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and functions $\mathcal{C}$, $\mathcal{D}$ from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{B}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters meet the following conditions:

- For every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{C}(\text{the root tree of } t) = \mathcal{F}(t)$,

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- Let n_1 be a symbol of $\mathcal{A}$ and let t_1 be a finite sequence of elements of $\mathrm{TS}(\mathcal{A})$. Suppose $n_1 \Rightarrow$ the roots of t_1. Let x be a finite sequence of elements of $\mathcal{B}$. If $x = \mathcal{C} \cdot t_1$, then $\mathcal{C}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, t_1, x)$,
- For every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{D}$(the root tree of t) $= \mathcal{F}(t)$,
- Let n_1 be a symbol of $\mathcal{A}$ and let t_1 be a finite sequence of elements of $\mathrm{TS}(\mathcal{A})$. Suppose $n_1 \Rightarrow$ the roots of t_1. Let x be a finite sequence of elements of $\mathcal{B}$. If $x = \mathcal{D} \cdot t_1$, then $\mathcal{D}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, t_1, x)$.

The following two propositions are true:

(2) Let S be a non void non empty many sorted signature, and let X be a many sorted set indexed by the carrier of S, and let o, b be arbitrary. Suppose $\langle o, b\rangle \in \mathrm{REL}(X)$. Then

(i) $o \in [\!:$ the operation symbols of S, $\{$the carrier of $S\}\,:\!]$, and

(ii) $b \in ([\!:$ the operation symbols of S, $\{$the carrier of $S\}\,:\!] \cup \bigcup \mathrm{coprod}(X))^*$.

(3) Let S be a non void non empty many sorted signature, and let X be a many sorted set indexed by the carrier of S, and let o be an operation symbol of S, and let b be a finite sequence. Suppose $\langle\langle o$, the carrier of $S\rangle, b\rangle \in \mathrm{REL}(X)$. Then

(i) $\mathrm{len}\, b = \mathrm{len}\, \mathrm{Arity}(o)$, and

(ii) for arbitrary x such that $x \in \mathrm{dom}\, b$ holds if $b(x) \in [\!:$ the operation symbols of S, $\{$the carrier of $S\}\,:\!]$, then for every operation symbol o_1 of S such that $\langle o_1$, the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 = \mathrm{Arity}(o)(x)$ and if $b(x) \in \bigcup \mathrm{coprod}(X)$, then $b(x) \in \mathrm{coprod}(\mathrm{Arity}(o)(x), X)$.

Let I be a non empty set and let M be a non-empty many sorted set indexed by I. Observe that $\mathrm{rng}\, M$ is non empty and has non empty elements.

Let D be a non empty set with non empty elements. Note that $\bigcup D$ is non empty.

Let I be a set. One can check that every many sorted set indexed by I which is empty is also pairwise disjoint.

Let I be a set. Observe that there exists a many sorted set indexed by I which is pairwise disjoint.

Let I be a non empty set, let X be a pairwise disjoint many sorted set indexed by I, let D be a non-empty many sorted set indexed by I, and let F be a many sorted function from X into D. The functor $\mathrm{Flatten}(F)$ yields a function from $\bigcup X$ into $\bigcup D$ and is defined by:

(Def.1) For every element i of I and for arbitrary x such that $x \in X(i)$ holds $(\mathrm{Flatten}(F))(x) = F(i)(x)$.

The following proposition is true

(4) Let I be a non empty set, and let X be a pairwise disjoint many sorted set indexed by I, and let D be a non-empty many sorted set indexed by I, and let F_1, F_2 be many sorted functions from X into D. If $\mathrm{Flatten}(F_1) = \mathrm{Flatten}(F_2)$, then $F_1 = F_2$.

Let S be a non empty many sorted signature and let A be an algebra over S. We say that A is pairwise disjoint if and only if:

(Def.2) The sorts of A is pairwise disjoint.

Let S be a non empty many sorted signature. The functor SingleAlg(S) yields a strict algebra over S and is defined by:

(Def.3) For arbitrary i such that $i \in$ the carrier of S holds (the sorts of SingleAlg$(S))(i) = \{i\}$.

Let S be a non empty many sorted signature. Note that there exists an algebra over S which is non-empty and pairwise disjoint.

Let S be a non empty many sorted signature. Observe that SingleAlg(S) is non-empty and pairwise disjoint.

Let S be a non empty many sorted signature and let A be a pairwise disjoint algebra over S. Observe that the sorts of A is pairwise disjoint.

The following proposition is true

(5) Let S be a non void non empty many sorted signature, and let o be an operation symbol of S, and let A_1 be a non-empty pairwise disjoint algebra over S, and let A_2 be a non-empty algebra over S, and let f be a many sorted function from A_1 into A_2, and let a be an element of $\mathrm{Args}(o, A_1)$. Then $\mathrm{Flatten}(f) \cdot a = f\#a$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Observe that FreeSorts(X) is pairwise disjoint.

The scheme *FreeSortUniq* deals with a non void non empty many sorted signature $\mathcal{A}$, non-empty many sorted sets $\mathcal{B}$, $\mathcal{C}$ indexed by the carrier of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\bigcup\mathcal{C}$, a ternary functor $\mathcal{G}$ yielding an element of $\bigcup\mathcal{C}$, and many sorted functions $\mathcal{D}$, $\mathcal{E}$ from FreeSorts$(\mathcal{B})$ into $\mathcal{C}$, and states that:

$$\mathcal{D} = \mathcal{E}$$

provided the following conditions are satisfied:

- Let o be an operation symbol of $\mathcal{A}$, and let t_1 be an element of $\mathrm{Args}(o, \mathrm{Free}(\mathcal{B}))$, and let x be a finite sequence of elements of $\bigcup\mathcal{C}$. If $x = \mathrm{Flatten}(\mathcal{D}) \cdot t_1$, then $\mathcal{D}(\text{the result sort of } o)((\mathrm{Den}(o, \mathrm{Free}(\mathcal{B})))(t_1)) = \mathcal{G}(o, t_1, x)$,
- For every sort symbol s of $\mathcal{A}$ and for arbitrary y such that $y \in \mathrm{FreeGenerator}(s, \mathcal{B})$ holds $\mathcal{D}(s)(y) = \mathcal{F}(y)$,
- Let o be an operation symbol of $\mathcal{A}$, and let t_1 be an element of $\mathrm{Args}(o, \mathrm{Free}(\mathcal{B}))$, and let x be a finite sequence of elements of $\bigcup\mathcal{C}$. If $x = \mathrm{Flatten}(\mathcal{E}) \cdot t_1$, then $\mathcal{E}(\text{the result sort of } o)((\mathrm{Den}(o, \mathrm{Free}(\mathcal{B})))(t_1)) = \mathcal{G}(o, t_1, x)$,
- For every sort symbol s of $\mathcal{A}$ and for arbitrary y such that $y \in \mathrm{FreeGenerator}(s, \mathcal{B})$ holds $\mathcal{E}(s)(y) = \mathcal{F}(y)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Note that Free(X) is non-empty.

Let S be a non void non empty many sorted signature, let o be an operation symbol of S, and let A be a non-empty algebra over S. Note that $\mathrm{Args}(o, A)$ is non empty and Result(o, A) is non empty.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Note that the sorts of Free(X) is pairwise disjoint.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. One can verify that Free(X) is pairwise disjoint.

The scheme *ExtFreeGen* deals with a non void non empty many sorted signature $\mathcal{A}$, a non-empty many sorted set $\mathcal{B}$ indexed by the carrier of $\mathcal{A}$, a non-empty algebra $\mathcal{C}$ over $\mathcal{A}$, many sorted functions $\mathcal{D}$, $\mathcal{E}$ from Free$(\mathcal{B})$ into $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

$\mathcal{D} = \mathcal{E}$

provided the following conditions are satisfied:

- $\mathcal{D}$ is a homomorphism of Free$(\mathcal{B})$ into $\mathcal{C}$,
- For every sort symbol s of $\mathcal{A}$ and for arbitrary x, y such that $y \in$ FreeGenerator$(s, \mathcal{B})$ holds $\mathcal{D}(s)(y) = x$ iff $\mathcal{P}[s, x, y]$,
- $\mathcal{E}$ is a homomorphism of Free$(\mathcal{B})$ into $\mathcal{C}$,
- For every sort symbol s of $\mathcal{A}$ and for arbitrary x, y such that $y \in$ FreeGenerator$(s, \mathcal{B})$ holds $\mathcal{E}(s)(y) = x$ iff $\mathcal{P}[s, x, y]$.

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