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Preliminaries to Circuits, II [1]

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Summary. This article is the second in a series of four articles (started with [20] and continued in [19,18]) about modelling circuits by many sorted algebras.

First, we introduce some additional terminology for many sorted signatures. The vertices of such signatures are divided into input vertices and inner vertices. A many sorted signature is called *circuit like* if each sort is a result sort of at most one operation. Next, we introduce some notions for many sorted algebras and many sorted free algebras. Free envelope of an algebra is a free algebra generated by the sorts of the algebra. Evaluation of an algebra is defined as a homomorphism from the free envelope of the algebra into the algebra. We define depth of elements of free many sorted algebras.

A many sorted signature is said to be monotonic if every finitely generated algebra over it is locally finite (finite in each sort). Monotonic signatures are used (see [19,18]) in modelling backbones of circuits without directed cycles.

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The papers [24], [28], [25], [1], [29], [12], [15], [7], [13], [5], [2], [4], [6], [3], [23], [17], [22], [11], [21], [9], [10], [8], [14], [26], [30], [16], [27], and [20] provide the notation and terminology for this paper.

1. Many Sorted Signatures

Let S be a many sorted signature. A vertex of S is an element of the carrier of S.

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Let S be a non empty many sorted signature.

The functor SortsWithConstants(S) yielding a subset of the carrier of S is defined as follows:

(Def.1) (i) SortsWithConstants(S) $= \{v : v$ ranges over sort symbols of S, v has constants$\}$ if S is non void,

(ii) SortsWithConstants(S) $= \emptyset$, otherwise.

Let G be a non empty many sorted signature. The functor InputVertices(G) yields a subset of the carrier of G and is defined by:

(Def.2) InputVertices(G) = (the carrier of G) $\setminus$ rng (the result sort of G).

The functor InnerVertices(G) yielding a subset of the carrier of G is defined by:

(Def.3) InnerVertices(G) = rng (the result sort of G).

Next we state several propositions:

(1) For every void non empty many sorted signature G holds InputVertices(G) = the carrier of G.

(2) Let G be a non void non empty many sorted signature and let v be a vertex of G. Suppose $v \in$ InputVertices(G). Then it is not true that there exists an operation symbol o of G such that the result sort of $o = v$.

(3) For every non empty many sorted signature G holds InputVertices(G) $\cup$ InnerVertices(G) = the carrier of G.

(4) For every non empty many sorted signature G holds InputVertices(G) misses InnerVertices(G).

(5) For every non empty many sorted signature G holds SortsWithConstants(G) $\subseteq$ InnerVertices(G).

(6) For every non empty many sorted signature G holds InputVertices(G) misses SortsWithConstants(G).

A non empty many sorted signature has input vertices if:

(Def.4) InputVertices(it) $\neq \emptyset$.

Let us note that there exists a non empty many sorted signature which is non void and has input vertices.

Let G be a non empty many sorted signature with input vertices. Note that InputVertices(G) is non empty.

Let G be a non void non empty many sorted signature. Then InnerVertices(G) is a non empty subset of the carrier of G.

Let S be a non empty many sorted signature and let M_1 be a non-empty algebra over S. A many sorted set indexed by InputVertices(S) is said to be an input assignment of M_1 if:

(Def.5) For every vertex v of S such that $v \in$ InputVertices(S) holds it(v) $\in$ (the sorts of M_1)(v).

Let S be a non empty many sorted signature. We say that S is circuit-like if and only if the condition (Def.6) is satisfied.

(Def.6) Let S' be a non void non empty many sorted signature. Suppose $S' = S$. Let o_1, o_2 be operation symbols of S'. If the result sort of $o_1 =$ the result sort of o_2, then $o_1 = o_2$.

Let us observe that every non empty many sorted signature which is void is also circuit-like.

Let us note that there exists a non empty many sorted signature which is non void circuit-like and strict.

Let I_1 be a circuit-like non void non empty many sorted signature and let v be a vertex of I_1. Let us assume that $v \in \mathrm{InnerVertices}(I_1)$. The action at v yielding an operation symbol of I_1 is defined as follows:

(Def.7) The result sort of the action at $v = v$.

2. Free Many Sorted Algebras

Next we state the proposition

(7) Let S be a non void non empty many sorted signature, and let A be an algebra over S, and let o be an operation symbol of S, and let p be a finite sequence. Suppose $\mathrm{len}\, p = \mathrm{len}\,\mathrm{Arity}(o)$ and for every natural number k such that $k \in \mathrm{dom}\, p$ holds $p(k) \in$ (the sorts of A)($\pi_k \mathrm{Arity}(o)$). Then $p \in \mathrm{Args}(o, A)$.

Let S be a non void non empty many sorted signature and let M_1 be a non-empty algebra over S. The functor FreeEnvelope(M_1) yielding a free strict non-empty algebra over S is defined as follows:

(Def.8) FreeEnvelope(M_1) = Free(the sorts of M_1).

One can prove the following proposition

(8) Let S be a non void non empty many sorted signature and let M_1 be a non-empty algebra over S. Then FreeGenerator(the sorts of M_1) is a free generator set of FreeEnvelope(M_1).

Let S be a non void non empty many sorted signature and let M_1 be a non-empty algebra over S. The functor Eval(M_1) yielding a many sorted function from FreeEnvelope(M_1) into M_1 is defined by the conditions (Def.9).

(Def.9) (i) Eval(M_1) is a homomorphism of FreeEnvelope(M_1) into M_1, and

(ii) for every sort symbol s of S and for arbitrary x, y such that $y \in$ FreeSort(the sorts of M_1, s) and $y =$ the root tree of $\langle x, s \rangle$ and $x \in$ (the sorts of M_1)(s) holds (Eval(M_1))(s)(y) $= x$.

One can prove the following proposition

(9) Let S be a non void non empty many sorted signature and let A be a non-empty algebra over S. Then the sorts of A is a generator set of A.

Let S be a non empty many sorted signature. An algebra over S is finitely-generated if:

(Def.10) (i) For every non void non empty many sorted signature S' such that $S' = S$ and for every algebra A over S' such that $A =$ it holds there exists generator set of A which is locally-finite if S is not void,

(ii) the sorts of it is locally-finite, otherwise.

Let S be a non empty many sorted signature. An algebra over S is locally-finite if:

(Def.11) The sorts of it is locally-finite.

Let S be a non empty many sorted signature. Observe that every non-empty algebra over S which is locally-finite is also finitely-generated.

Let S be a non empty many sorted signature. The trivial algebra of S yields a strict algebra over S and is defined by:

(Def.12) The sorts of the trivial algebra of $S =$ (the carrier of S) $\longmapsto \{0\}$.

Let S be a non empty many sorted signature. Observe that there exists an algebra over S which is locally-finite non-empty and strict.

A non empty many sorted signature is monotonic if:

(Def.13) Every finitely-generated non-empty algebra over it is locally-finite.

One can verify that there exists a non empty many sorted signature which is non void finite monotonic and circuit-like.

The following propositions are true:

(10) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set indexed by the carrier of S, and let v be a sort symbol of S. Then every element of the sorts of $\mathrm{Free}(X)(v)$ is a finite decorated tree.

(11) Let S be a non void non empty many sorted signature and let X be a non-empty locally-finite many sorted set indexed by the carrier of S. Then $\mathrm{Free}(X)$ is finitely-generated.

(12) Let S be a non void non empty many sorted signature, and let A be a non-empty algebra over S, and let v be a vertex of S, and let e be an element of (the sorts of $\mathrm{FreeEnvelope}(A)$)(v). Suppose $v \in \mathrm{InputVertices}(S)$. Then there exists an element x of (the sorts of A)(v) such that $e =$ the root tree of $\langle x, v\rangle$.

(13) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set indexed by the carrier of S, and let o be an operation symbol of S, and let p be a decorated tree yielding finite sequence. Suppose $\langle o$, the carrier of $S\rangle$-tree$(p) \in$ (the sorts of $\mathrm{Free}(X)$)(the result sort of o). Then $\mathrm{len}\, p = \mathrm{len}\,\mathrm{Arity}(o)$.

(14) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set indexed by the carrier of S, and let o be an operation symbol of S, and let p be a decorated tree yielding finite sequence. Suppose $\langle o$, the carrier of $S\rangle$-tree$(p) \in$ (the sorts of $\mathrm{Free}(X)$)(the result sort of o). Let i be a natural number. If $i \in \mathrm{dom}\,\mathrm{Arity}(o)$, then $p(i) \in$ (the sorts of $\mathrm{Free}(X)$)$(\mathrm{Arity}(o)(i))$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let v be a vertex of S. One can check that every element of the sorts of $\mathrm{Free}(X)(v)$ is finite non empty function-like and relation-like.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let v be a vertex of S. Note that there exists an element of the sorts of $\mathrm{Free}(X)(v)$ which is function-like and relation-like.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let v be a vertex of S. Observe that every function-like relation-like element of the sorts of $\mathrm{Free}(X)(v)$ is decorated tree-like.

Let I_1 be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of I_1, and let v be a vertex of I_1. Observe that there exists an element of the sorts of $\mathrm{Free}(X)(v)$ which is finite.

We now state the proposition

(15) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set indexed by the carrier of S, and let v be a vertex of S, and let o be an operation symbol of S, and let e be an element of (the sorts of $\mathrm{Free}(X)$)(v). Suppose $v \in \mathrm{InnerVertices}(S)$ and $e(\varepsilon) = \langle o, \text{the carrier of } S\rangle$. Then there exists a decorated tree yielding finite sequence p such that $\mathrm{len}\, p = \mathrm{len}\, \mathrm{Arity}(o)$ and for every natural number i such that $i \in \mathrm{dom}\, p$ holds $p(i) \in (\text{the sorts of } \mathrm{Free}(X))(\mathrm{Arity}(o)(i))$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, let v be a sort symbol of S, and let e be an element of (the sorts of $\mathrm{Free}(X)$)(v). The functor $\mathrm{depth}(e)$ yielding a natural number is defined by:

(Def.14) There exists a finite decorated tree d_1 and there exists a finite tree t such that $d_1 = e$ and $t = \mathrm{dom}\, d_1$ and $\mathrm{depth}(e) = \mathrm{height}\, t$.

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