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Terms Over Many Sorted Universal Algebra [1]

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Summary. Pure terms (without constants) over a signature of many sorted universal algebra and terms with constants from algebra are introduced. Facts on evaluation of a term in some valuation are proved.

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The articles [19], [22], [2], [20], [23], [11], [9], [12], [14], [3], [5], [6], [21], [1], [13], [7], [4], [8], [18], [17], [10], [15], and [16] provide the terminology and notation for this paper.

1. Terms over a Signature and over an Algebra

Let I be a non empty set, let X be a non-empty many sorted set indexed by I, and let i be an element of I. Note that $X(i)$ is non empty.

In the sequel S will be a non void non empty many sorted signature and V will be a non-empty many sorted set indexed by the carrier of S.

Let us consider S, V. The functor S-Terms(V) yielding a non empty subset of FinTrees(the carrier of DTConMSA(V)) is defined as follows:

(Def.1) S-Terms(V) = TS(DTConMSA(V)).

Let us consider S, V. A term of S over V is an element of S-Terms(V).

In the sequel A denotes an algebra over S and t denotes a term of S over V.

Let us consider S, V and let o be an operation symbol of S. Then Sym(o, V) is a nonterminal of DTConMSA(V).

Let us consider S, V and let s_1 be a nonterminal of DTConMSA(V). A finite sequence of elements of S-Terms(V) is called an argument sequence of s_1 if:

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(Def.2) It is a subtree sequence joinable by s_1.

We now state the proposition

(1) Let o be an operation symbol of S and let a be a finite sequence. Then $\langle o$, the carrier of $S\rangle$-tree$(a) \in S$-Terms(V) and a is decorated tree yielding if and only if a is an argument sequence of $\mathrm{Sym}(o, V)$.

The scheme *TermInd* concerns a non void non empty many sorted signature $\mathcal{A}$, a non-empty many sorted set $\mathcal{B}$ indexed by the carrier of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

For every term t of $\mathcal{A}$ over $\mathcal{B}$ holds $\mathcal{P}[t]$

provided the parameters satisfy the following conditions:

- For every sort symbol s of $\mathcal{A}$ and for every element v of $\mathcal{B}(s)$ holds $\mathcal{P}$[the root tree of $\langle v, s\rangle$],
- Let o be an operation symbol of $\mathcal{A}$ and let p be an argument sequence of $\mathrm{Sym}(o, \mathcal{B})$. Suppose that for every term t of $\mathcal{A}$ over $\mathcal{B}$ such that $t \in \mathrm{rng}\, p$ holds $\mathcal{P}[t]$. Then $\mathcal{P}[\langle o$, the carrier of $\mathcal{A}\rangle$-tree$(p)]$.

Let us consider S, A, V. A term of A over V is a term of S over (the sorts of $A) \cup (V)$.

Let us consider S, A, V and let o be an operation symbol of S. An argument sequence of o, A, and V is an argument sequence of $\mathrm{Sym}(o,$ (the sorts of $A) \cup (V))$.

The scheme *CTermInd* concerns a non void non empty many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a non-empty many sorted set $\mathcal{C}$ indexed by the carrier of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

For every term t of $\mathcal{B}$ over $\mathcal{C}$ holds $\mathcal{P}[t]$

provided the following requirements are met:

- For every sort symbol s of $\mathcal{A}$ and for every element x of (the sorts of $\mathcal{B})(s)$ holds $\mathcal{P}$[the root tree of $\langle x, s\rangle$],
- For every sort symbol s of $\mathcal{A}$ and for every element v of $\mathcal{C}(s)$ holds $\mathcal{P}$[the root tree of $\langle v, s\rangle$],
- Let o be an operation symbol of $\mathcal{A}$ and let p be an argument sequence of o, $\mathcal{B}$, and $\mathcal{C}$. Suppose that for every term t of $\mathcal{B}$ over $\mathcal{C}$ such that $t \in \mathrm{rng}\, p$ holds $\mathcal{P}[t]$. Then $\mathcal{P}[\mathrm{Sym}(o,$ (the sorts of $\mathcal{B}) \cup \mathcal{C})$-tree$(p)]$.

Let us consider S, V, t and let p be a node of t. Then $t(p)$ is a symbol of DTConMSA(V).

Let us consider S, V. Observe that every term of S over V is finite.

Next we state several propositions:

(2) (i) There exists a sort symbol s of S and there exists an element v of $V(s)$ such that $t(\varepsilon) = \langle v, s\rangle$, or

(ii) $t(\varepsilon) \in [\!:$ the operation symbols of S, {the carrier of S} $:\!]$.

(3) Let t be a term of A over V. Then

(i) there exists a sort symbol s of S and there exists a set x such that $x \in$ (the sorts of $A)(s)$ and $t(\varepsilon) = \langle x, s\rangle$, or

(ii) there exists a sort symbol s of S and there exists an element v of $V(s)$ such that $t(\varepsilon) = \langle v, s\rangle$, or

(iii) $t(\varepsilon) \in [\!:$ the operation symbols of S, $\{$the carrier of $S\}$ $:\!]$.

(4) For every sort symbol s of S and for every element v of $V(s)$ holds the root tree of $\langle v, s\rangle$ is a term of S over V.

(5) For every sort symbol s of S and for every element v of $V(s)$ such that $t(\varepsilon) = \langle v, s\rangle$ holds $t =$ the root tree of $\langle v, s\rangle$.

(6) Let s be a sort symbol of S and let x be a set. Suppose $x \in$ (the sorts of A)(s). Then the root tree of $\langle x, s\rangle$ is a term of A over V.

(7) Let t be a term of A over V, and let s be a sort symbol of S, and let x be a set. If $x \in$ (the sorts of A)(s) and $t(\varepsilon) = \langle x, s\rangle$, then $t =$ the root tree of $\langle x, s\rangle$.

(8) For every sort symbol s of S and for every element v of $V(s)$ holds the root tree of $\langle v, s\rangle$ is a term of A over V.

(9) Let t be a term of A over V, and let s be a sort symbol of S, and let v be an element of $V(s)$. If $t(\varepsilon) = \langle v, s\rangle$, then $t =$ the root tree of $\langle v, s\rangle$.

(10) Let o be an operation symbol of S. Suppose $t(\varepsilon) = \langle o$, the carrier of $S\rangle$. Then there exists an argument sequence a of Sym(o, V) such that $t = \langle o$, the carrier of $S\rangle$-tree(a).

Let us consider S, let A be a non-empty algebra over S, let us consider V, let s be a sort symbol of S, and let x be an element of (the sorts of A)(s). The functor $x_{A,V}$ yielding a term of A over V is defined as follows:

(Def.3) $x_{A,V} =$ the root tree of $\langle x, s\rangle$.

Let us consider S, A, V, let s be a sort symbol of S, and let v be an element of $V(s)$. The functor v_A yields a term of A over V and is defined as follows:

(Def.4) $v_A =$ the root tree of $\langle v, s\rangle$.

Let us consider S, V, let s_1 be a nonterminal of DTConMSA(V), and let p be an argument sequence of s_1. Then s_1-tree(p) is a term of S over V.

The scheme *TermInd2* concerns a non void non empty many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a non-empty many sorted set $\mathcal{C}$ indexed by the carrier of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

For every term t of $\mathcal{B}$ over $\mathcal{C}$ holds $\mathcal{P}[t]$

provided the following conditions are satisfied:

- For every sort symbol s of $\mathcal{A}$ and for every element x of (the sorts of $\mathcal{B}$)(s) holds $\mathcal{P}[x_{\mathcal{B},\mathcal{C}}]$,
- For every sort symbol s of $\mathcal{A}$ and for every element v of $\mathcal{C}(s)$ holds $\mathcal{P}[v_{\mathcal{B}}]$,
- Let o be an operation symbol of $\mathcal{A}$ and let p be an argument sequence of Sym(o, (the sorts of $\mathcal{B}$) $\cup\, \mathcal{C}$). Suppose that for every term t of $\mathcal{B}$ over $\mathcal{C}$ such that $t \in \operatorname{rng} p$ holds $\mathcal{P}[t]$. Then $\mathcal{P}$[Sym(o, (the sorts of $\mathcal{B}$) $\cup\, \mathcal{C}$)-tree(p)].

2. Sort of a Term

One can prove the following three propositions:

(11) For every term t of S over V there exists a sort symbol s of S such that $t \in \mathrm{FreeSort}(V, s)$.

(12) For every sort symbol s of S holds $\mathrm{FreeSort}(V, s) \subseteq S\text{-Terms}(V)$.

(13) $S\text{-Terms}(V) = \bigcup \mathrm{FreeSorts}(V)$.

Let us consider S, V, t. The sort of t yields a sort symbol of S and is defined by:

(Def.5) $t \in \mathrm{FreeSort}(V, \text{the sort of } t)$.

One can prove the following propositions:

(14) Let s be a sort symbol of S and let v be an element of $V(s)$. If $t =$ the root tree of $\langle v, s \rangle$, then the sort of $t = s$.

(15) Let t be a term of A over V, and let s be a sort symbol of S, and let x be a set. Suppose $x \in$ (the sorts of A)(s) and $t =$ the root tree of $\langle x, s \rangle$. Then the sort of $t = s$.

(16) Let t be a term of A over V, and let s be a sort symbol of S, and let v be an element of $V(s)$. If $t =$ the root tree of $\langle v, s \rangle$, then the sort of $t = s$.

(17) Let o be an operation symbol of S. Suppose $t(\varepsilon) = \langle o$, the carrier of $S \rangle$. Then the sort of $t =$ the result sort of o.

(18) Let A be a non-empty algebra over S, and let s be a sort symbol of S, and let x be an element of (the sorts of A)(s). Then the sort of $x_{A,V} = s$.

(19) For every sort symbol s of S and for every element v of $V(s)$ holds the sort of $v_A = s$.

(20) Let o be an operation symbol of S and let p be an argument sequence of $\mathrm{Sym}(o, V)$. Then the sort of ($\mathrm{Sym}(o, V)$-tree(p) **qua** term of S over V) = the result sort of o.

3. Argument Sequence

We now state several propositions:

(21) Let o be an operation symbol of S and let a be a finite sequence of elements of $S\text{-Terms}(V)$. Then a is an argument sequence of $\mathrm{Sym}(o, V)$ if and only if $\mathrm{Sym}(o, V) \Rightarrow$ the roots of a.

(22) Let o be an operation symbol of S and let a be an argument sequence of $\mathrm{Sym}(o, V)$. Then $\mathrm{len}\, a = \mathrm{len}\, \mathrm{Arity}(o)$ and $\mathrm{dom}\, a = \mathrm{dom}\, \mathrm{Arity}(o)$ and for every natural number i such that $i \in \mathrm{dom}\, a$ holds $a(i)$ is a term of S over V.

(23) Let o be an operation symbol of S, and let a be an argument sequence of $\mathrm{Sym}(o, V)$, and let i be a natural number. Suppose $i \in \mathrm{dom}\, a$. Let t be a term of S over V. Suppose $t = a(i)$. Then

(i) $t = \pi_i(a$ **qua** finite sequence of elements of S-Terms(V) **qua** non empty set),

(ii) the sort of $t = \mathrm{Arity}(o)(i)$, and

(iii) the sort of $t = \pi_i\, \mathrm{Arity}(o)$.

(24) Let o be an operation symbol of S and let a be a finite sequence. Suppose that

(i) $\mathrm{len}\, a = \mathrm{len}\, \mathrm{Arity}(o)$ or $\mathrm{dom}\, a = \mathrm{dom}\, \mathrm{Arity}(o)$, and

(ii) for every natural number i such that $i \in \mathrm{dom}\, a$ there exists a term t of S over V such that $t = a(i)$ and the sort of $t = \mathrm{Arity}(o)(i)$ or for every natural number i such that $i \in \mathrm{dom}\, a$ there exists a term t of S over V such that $t = a(i)$ and the sort of $t = \pi_i\, \mathrm{Arity}(o)$.

Then a is an argument sequence of $\mathrm{Sym}(o, V)$.

(25) Let o be an operation symbol of S and let a be a finite sequence of elements of S-Terms(V). Suppose that

(i) $\mathrm{len}\, a = \mathrm{len}\, \mathrm{Arity}(o)$ or $\mathrm{dom}\, a = \mathrm{dom}\, \mathrm{Arity}(o)$, and

(ii) for every natural number i such that $i \in \mathrm{dom}\, a$ and for every term t of S over V such that $t = a(i)$ holds the sort of $t = \mathrm{Arity}(o)(i)$ or for every natural number i such that $i \in \mathrm{dom}\, a$ and for every term t of S over V such that $t = a(i)$ holds the sort of $t = \pi_i\, \mathrm{Arity}(o)$.

Then a is an argument sequence of $\mathrm{Sym}(o, V)$.

(26) Let S be a non void non empty many sorted signature and let V_1, V_2 be non-empty many sorted sets indexed by the carrier of S. If $V_1 \subseteq V_2$, then every term of S over V_1 is a term of S over V_2.

(27) Let S be a non void non empty many sorted signature, and let A be an algebra over S, and let V be a non-empty many sorted set indexed by the carrier of S. Then every term of S over V is a term of A over V.

4. Compound Terms

Let S be a non void non empty many sorted signature and let V be a non-empty many sorted set indexed by the carrier of S. A term of S over V is said to be a compound term of S over V if:

(Def.6) $\mathrm{It}(\varepsilon) \in [\!:$ the operation symbols of S, $\{$the carrier of $S\}\,:\!]$.

Let S be a non void non empty many sorted signature and let V be a non-empty many sorted set indexed by the carrier of S. A non empty subset of S-Terms(V) is called a set with a compound term of S over V if:

(Def.7) There exists a compound term t of S over V such that $t \in$ it.

Next we state two propositions:

(28) If t is not root, then t is a compound term of S over V.

(29) For every node p of t holds $t \restriction p$ is a term of S over V.

Let S be a non void non empty many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, let t be a term of S over V, and let p be a node of t. Then $t \restriction p$ is a term of S over V.

5. Evaluation of Terms

Let S be a non void non empty many sorted signature and let A be an algebra over S. A non-empty many sorted set indexed by the carrier of S is said to be a variables family of A if:

(Def.8) It misses the sorts of A.

We now state the proposition

(30) Let V be a variables family of A, and let s be a sort symbol of S, and let x be a set. If $x \in$ (the sorts of A)(s), then for every element v of $V(s)$ holds $x \neq v$.

Let S be a non void non empty many sorted signature, let A be a non-empty algebra over S, let V be a non-empty many sorted set indexed by the carrier of S, let t be a term of A over V, let f be a many sorted function from V into the sorts of A, and let v_1 be a finite decorated tree. We say that v_1 is an evaluation of t w.r.t. f if and only if the conditions (Def.9) are satisfied.

(Def.9) (i) $\operatorname{dom} v_1 = \operatorname{dom} t$, and

(ii) for every node p of v_1 holds for every sort symbol s of S and for every element v of $V(s)$ such that $t(p) = \langle v, s\rangle$ holds $v_1(p) = f(s)(v)$ and for every sort symbol s of S and for every element x of (the sorts of A)(s) such that $t(p) = \langle x, s\rangle$ holds $v_1(p) = x$ and for every operation symbol o of S such that $t(p) = \langle o, \text{the carrier of } S\rangle$ holds $v_1(p) = (\mathrm{Den}(o, A))(\mathrm{succ}(v_1, p))$.

For simplicity we follow the rules: S will be a non void non empty many sorted signature, A will be a non-empty algebra over S, V will be a variables family of A, t will be a term of A over V, and f will be a many sorted function from V into the sorts of A.

We now state several propositions:

(31) Let s be a sort symbol of S and let x be an element of (the sorts of A)(s). Suppose $t =$ the root tree of $\langle x, s\rangle$. Then the root tree of x is an evaluation of t w.r.t. f.

(32) Let s be a sort symbol of S and let v be an element of $V(s)$. Suppose $t =$ the root tree of $\langle v, s\rangle$. Then the root tree of $f(s)(v)$ is an evaluation of t w.r.t. f.

(33) Let o be an operation symbol of S, and let p be an argument sequence of o, A, and V, and let q be a decorated tree yielding finite sequence. Suppose that

(i) $\operatorname{len} q = \operatorname{len} p$, and

(ii) for every natural number i and for every term t of A over V such that $i \in \operatorname{dom} p$ and $t = p(i)$ there exists a finite decorated tree v_1 such that $v_1 = q(i)$ and v_1 is an evaluation of t w.r.t. f.

Then there exists a finite decorated tree v_1 such that $v_1 = (\mathrm{Den}(o, A))$(the roots of q)-tree(q) and v_1 is an evaluation of Sym(o, (the sorts of A) $\cup$ (V))-tree(p) **qua** term of A over V w.r.t. f.

(34) Let t be a term of A over V and let e be a finite decorated tree. Suppose e is an evaluation of t w.r.t. f. Let p be a node of t and let n be a node of e. If $n = p$, then $e \restriction n$ is an evaluation of $t \restriction p$ w.r.t. f.

(35) Let o be an operation symbol of S, and let p be an argument sequence of o, A, and V, and let v_1 be a finite decorated tree. Suppose v_1 is an evaluation of Sym(o, (the sorts of A) $\cup$ (V))-tree(p) **qua** term of A over V w.r.t. f. Then there exists a decorated tree yielding finite sequence q such that

(i) $\operatorname{len} q = \operatorname{len} p$,

(ii) $v_1 = (\mathrm{Den}(o, A))$(the roots of q)-tree(q), and

(iii) for every natural number i and for every term t of A over V such that $i \in \operatorname{dom} p$ and $t = p(i)$ there exists a finite decorated tree v_1 such that $v_1 = q(i)$ and v_1 is an evaluation of t w.r.t. f.

(36) There exists finite decorated tree which is an evaluation of t w.r.t. f.

(37) Let e_1, e_2 be finite decorated trees. Suppose e_1 is an evaluation of t w.r.t. f and e_2 is an evaluation of t w.r.t. f. Then $e_1 = e_2$.

(38) Let v_1 be a finite decorated tree. Suppose v_1 is an evaluation of t w.r.t. f. Then $v_1(\varepsilon) \in$ (the sorts of A)(the sort of t).

Let S be a non void non empty many sorted signature, let A be a non-empty algebra over S, let V be a variables family of A, let t be a term of A over V, and let f be a many sorted function from V into the sorts of A. The functor $t\ {}^{@}\ f$ yields an element of (the sorts of A)(the sort of t) and is defined as follows:

(Def.10) There exists a finite decorated tree v_1 such that v_1 is an evaluation of t w.r.t. f and $t\ {}^{@}\ f = v_1(\varepsilon)$.

In the sequel t denotes a term of A over V.

We now state several propositions:

(39) For every finite decorated tree v_1 such that v_1 is an evaluation of t w.r.t. f holds $t\ {}^{@}\ f = v_1(\varepsilon)$.

(40) Let v_1 be a finite decorated tree. Suppose v_1 is an evaluation of t w.r.t. f. Let p be a node of t. Then $v_1(p) = t \restriction p\ {}^{@}\ f$.

(41) For every sort symbol s of S and for every element x of (the sorts of A)(s) holds $x_{A,V}\ {}^{@}\ f = x$.

(42) For every sort symbol s of S and for every element v of $V(s)$ holds $v_A\ {}^{@}\ f = f(s)(v)$.

(43) Let o be an operation symbol of S, and let p be an argument sequence of o, A, and V, and let q be a finite sequence. Suppose that

(i) $\operatorname{len} q = \operatorname{len} p$, and

(ii) for every natural number i such that $i \in \operatorname{dom} p$ and for every term t of A over V such that $t = p(i)$ holds $q(i) = t\,^{@}\,f$.

Then (Sym(o, (the sorts of A)∪(V))-tree(p) **qua** term of A over V)$\,^{@}(f) =$ (Den(o, A))(q).

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