FORMALIZED MATHEMATICS
Volume 5, Number **2**, 1996
Warsaw University - Białystok

Special Polygons

Czesław Byliński
Warsaw University
Białystok

Yatsuka Nakamura
Shinshu University
Nagano

MML Identifier: `SPPOL_2`.

The papers [22], [26], [21], [25], [13], [1], [14], [27], [4], [5], [2], [23], [3], [10], [24], [19], [15], [18], [7], [9], [8], [20], [11], [12], [17], [16], and [6] provide the notation and terminology for this paper.

1. Segments in $\mathcal{E}^2_{\rm T}$

For simplicity we adopt the following convention: P, P_1, P_2 will be subsets of the carrier of $\mathcal{E}^2_{\rm T}$, f, f_1, f_2, g will be finite sequences of elements of $\mathcal{E}^2_{\rm T}$, p, p_1, p_2, q, q_1, q_2 will be points of $\mathcal{E}^2_{\rm T}$, r_1, r_2, r'_1, r'_2 will be real numbers, and i, j, k, n will be natural numbers.

Next we state a number of propositions:

(1) If $[r_1, r_2] = [r'_1, r'_2]$, then $r_1 = r'_1$ and $r_2 = r'_2$.

(2) If $i + j = \operatorname{len} f$, then $\mathcal{L}(f, i) = \mathcal{L}(\operatorname{Rev}(f), j)$.

(3) If $i + 1 \leq \operatorname{len}(f \restriction n)$, then $\mathcal{L}(f \restriction n, i) = \mathcal{L}(f, i)$.

(4) If $n \leq \operatorname{len} f$ and $1 \leq i$, then $\mathcal{L}(f_{\lfloor n}, i) = \mathcal{L}(f, n + i)$.

(5) If $1 \leq i$ and $i + 1 \leq \operatorname{len} f - n$, then $\mathcal{L}(f_{\lfloor n}, i) = \mathcal{L}(f, n + i)$.

(6) If $i + 1 \leq \operatorname{len} f$, then $\mathcal{L}(f \frown g, i) = \mathcal{L}(f, i)$.

(7) If $1 \leq i$, then $\mathcal{L}(f \frown g, \operatorname{len} f + i) = \mathcal{L}(g, i)$.

(8) If f is non empty and g is non empty, then $\mathcal{L}(f \frown g, \operatorname{len} f) = \mathcal{L}(\pi_{\operatorname{len} f} f, \pi_1 g)$.

(9) If $i + 1 \leq \operatorname{len}(f -: p)$, then $\mathcal{L}(f -: p, i) = \mathcal{L}(f, i)$.

(10) If $p \in \operatorname{rng} f$ and $1 \leq i + 1$, then $\mathcal{L}(f :- p, i + 1) = \mathcal{L}(f, i + p \leftarrow f)$.

(11) $\widetilde{\mathcal{L}}(\varepsilon_{(\text{the carrier of } \mathcal{E}^2_{\rm T})}) = \emptyset$.

(12) $\widetilde{\mathcal{L}}(\langle p \rangle) = \emptyset$.

ISSN 1426–2630

(13) If $p \in \widetilde{\mathcal{L}}(f)$, then there exists i such that $1 \leq i$ and $i+1 \leq \operatorname{len} f$ and $p \in \mathcal{L}(f,i)$.

(14) If $p \in \widetilde{\mathcal{L}}(f)$, then there exists i such that $1 \leq i$ and $i+1 \leq \operatorname{len} f$ and $p \in \mathcal{L}(\pi_i f, \pi_{i+1} f)$.

(15) If $1 \leq i$ and $i+1 \leq \operatorname{len} f$ and $p \in \mathcal{L}(\pi_i f, \pi_{i+1} f)$, then $p \in \widetilde{\mathcal{L}}(f)$.

(16) If $1 \leq i$ and $i+1 \leq \operatorname{len} f$, then $\mathcal{L}(\pi_i f, \pi_{i+1} f) \subseteq \widetilde{\mathcal{L}}(f)$.

(17) If $p \in \mathcal{L}(f,i)$, then $p \in \widetilde{\mathcal{L}}(f)$.

(18) If $\operatorname{len} f \geq 2$, then $\operatorname{rng} f \subseteq \widetilde{\mathcal{L}}(f)$.

(19) If f is non empty, then $\widetilde{\mathcal{L}}(f \frown \langle p \rangle) = \widetilde{\mathcal{L}}(f) \cup \mathcal{L}(\pi_{\operatorname{len} f} f, p)$.

(20) If f is non empty, then $\widetilde{\mathcal{L}}(\langle p \rangle \frown f) = \mathcal{L}(p, \pi_1 f) \cup \widetilde{\mathcal{L}}(f)$.

(21) $\widetilde{\mathcal{L}}(\langle p, q \rangle) = \mathcal{L}(p,q)$.

(22) $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(\operatorname{Rev}(f))$.

(23) If f_1 is non empty and f_2 is non empty, then $\widetilde{\mathcal{L}}(f_1 \frown f_2) = \widetilde{\mathcal{L}}(f_1) \cup \mathcal{L}(\pi_{\operatorname{len} f_1} f_1, \pi_1 f_2) \cup \widetilde{\mathcal{L}}(f_2)$.

(25)[1] If $q \in \operatorname{rng} f$, then $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(f -: q) \cup \widetilde{\mathcal{L}}(f :- q)$.

(26) If $p \in \mathcal{L}(f,n)$, then $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(\operatorname{Ins}(f,n,p))$.

2. Special Sequences in $\mathcal{E}^2_{\mathrm{T}}$

One can verify the following observations:

* there exists a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$
* every finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is one-to-one unfolded s.n.c. special and non trivial,
* every finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is one-to-one unfolded s.n.c. special and non trivial has and
* every finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is non empty.

Let us note that there exists a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is one-to-one unfolded s.n.c. special and non trivial.

We now state the proposition

(27) If $\operatorname{len} f \leq 2$, then f is unfolded.

Let f be an unfolded finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider n. Note that $f \restriction n$ is unfolded and $f_{\lfloor n}$ is unfolded.

One can prove the following proposition

(28) If $p \in \operatorname{rng} f$ and f is unfolded, then $f :- p$ is unfolded.

Let f be an unfolded finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider p. Observe that $f -: p$ is unfolded.

Next we state several propositions:

[1]The proposition (24) has been removed.

(29) If f is unfolded, then $\mathrm{Rev}(f)$ is unfolded.

(30) If g is unfolded and $\mathcal{L}(p, \pi_1 g) \cap \mathcal{L}(g, 1) = \{\pi_1 g\}$, then $\langle p \rangle \frown g$ is unfolded.

(31) If f is unfolded and $k+1 = \mathrm{len}\, f$ and $\mathcal{L}(f, k) \cap \mathcal{L}(\pi_{\mathrm{len}\, f} f, p) = \{\pi_{\mathrm{len}\, f} f\}$, then $f \frown \langle p \rangle$ is unfolded.

(32) Suppose f is unfolded and g is unfolded and $k+1 = \mathrm{len}\, f$ and $\mathcal{L}(f, k) \cap \mathcal{L}(\pi_{\mathrm{len}\, f} f, \pi_1 g) = \{\pi_{\mathrm{len}\, f} f\}$ and $\mathcal{L}(\pi_{\mathrm{len}\, f} f, \pi_1 g) \cap \mathcal{L}(g, 1) = \{\pi_1 g\}$. Then $f \frown g$ is unfolded.

(33) If f is unfolded and $p \in \mathcal{L}(f, n)$, then $\mathrm{Ins}(f, n, p)$ is unfolded.

(34) If $\mathrm{len}\, f \leq 2$, then f is s.n.c..

Let f be a s.n.c. finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider n. Observe that $f \restriction n$ is s.n.c. and $f_{\lfloor n}$ is s.n.c..

Let f be a s.n.c. finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider p. Note that $f -: p$ is s.n.c..

We now state four propositions:

(35) If $p \in \mathrm{rng}\, f$ and f is s.n.c., then $f :- p$ is s.n.c..

(36) If f is s.n.c., then $\mathrm{Rev}(f)$ is s.n.c..

(37) Suppose that
 (i) f is s.n.c.,
 (ii) g is s.n.c.,
 (iii) $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \emptyset$,
 (iv) for every i such that $1 \leq i$ and $i+2 \leq \mathrm{len}\, f$ holds $\mathcal{L}(f, i) \cap \mathcal{L}(\pi_{\mathrm{len}\, f} f, \pi_1 g) = \emptyset$, and
 (v) for every i such that $2 \leq i$ and $i+1 \leq \mathrm{len}\, g$ holds $\mathcal{L}(g, i) \cap \mathcal{L}(\pi_{\mathrm{len}\, f} f, \pi_1 g) = \emptyset$.

 Then $f \frown g$ is s.n.c..

(38) If f is unfolded and s.n.c. and $p \in \mathcal{L}(f, n)$ and $p \notin \mathrm{rng}\, f$, then $\mathrm{Ins}(f, n, p)$ is s.n.c..

Let us observe that $\varepsilon_{(\text{the carrier of } \mathcal{E}^2_{\mathrm{T}})}$ is special.

Next we state two propositions:

(39) $\langle p \rangle$ is special.

(40) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $\langle p, q \rangle$ is special.

Let f be a special finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider n. Note that $f \restriction n$ is special and $f_{\lfloor n}$ is special.

We now state the proposition

(41) If $p \in \mathrm{rng}\, f$ and f is special, then $f :- p$ is special.

Let f be a special finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider p. Observe that $f -: p$ is special.

The following four propositions are true:

(42) If f is special, then $\mathrm{Rev}(f)$ is special.

(44)[2] If f is special and $p \in \mathcal{L}(f, n)$, then $\mathrm{Ins}(f, n, p)$ is special.

[2]The proposition (43) has been removed.

(45) If $q \in \operatorname{rng} f$ and $1 \neq q \leftarrow\!\!\!\!\!\!\!\shortmid\; f$ and $q \leftarrow\!\!\!\!\!\!\!\shortmid\; f \neq \operatorname{len} f$ and f is unfolded and s.n.c., then $\widetilde{\mathcal{L}}(f -: q) \cap \widetilde{\mathcal{L}}(f :- q) = \{q\}$.

(46) If $p \neq q$ and if $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $\langle p, q \rangle$

a S-sequence in $\mathbb{R}^2$ is a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$.

The following propositions are true:

(47) For every S-sequence f in $\mathbb{R}^2$ holds $\operatorname{Rev}(f)$

(48) For every S-sequence f in $\mathbb{R}^2$ such that $i \in \operatorname{dom} f$ holds $\pi_i f \in \widetilde{\mathcal{L}}(f)$.

(49) If $p \neq q$ and if $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $\mathcal{L}(p, q)$

(50) For every S-sequence f in $\mathbb{R}^2$ such that $p \in \operatorname{rng} f$ and $p \leftarrow\!\!\!\!\!\!\!\shortmid\; f \neq 1$ holds $f -: p$

(51) For every S-sequence f in $\mathbb{R}^2$ such that $p \in \operatorname{rng} f$ and $p \leftarrow\!\!\!\!\!\!\!\shortmid\; f \neq \operatorname{len} f$ holds $f :- p$

(52) For every S-sequence f in $\mathbb{R}^2$ such that $p \in \mathcal{L}(f, i)$ and $p \notin \operatorname{rng} f$ holds $\operatorname{Ins}(f, i, p)$

3. Special Polygons in $\mathcal{E}^2_{\mathrm{T}}$

Let us mention that there exists a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and every subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is non empty.

The following proposition is true

(53) If P is a special polygonal arc joining p_1 and p_2, then P is a special polygonal arc joining p_2 and p_1.

Let us consider p_1, p_2, P. We say that p_1 and p_2 split P if and only if the conditions (Def.1) are satisfied.

(Def.1) (i) $p_1 \neq p_2$, and

(ii) there exist S-sequences f_1, f_2 in $\mathbb{R}^2$ such that $p_1 = \pi_1 f_1$ and $p_1 = \pi_1 f_2$ and $p_2 = \pi_{\operatorname{len} f_1} f_1$ and $p_2 = \pi_{\operatorname{len} f_2} f_2$ and $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) = \{p_1, p_2\}$ and $P = \widetilde{\mathcal{L}}(f_1) \cup \widetilde{\mathcal{L}}(f_2)$.

We now state four propositions:

(54) If p_1 and p_2 split P, then p_2 and p_1 split P.

(55) If p_1 and p_2 split P and $q \in P$ and $q \neq p_1$, then p_1 and q split P.

(56) If p_1 and p_2 split P and $q \in P$ and $q \neq p_2$, then q and p_2 split P.

(57) If p_1 and p_2 split P and $q_1 \in P$ and $q_2 \in P$ and $q_1 \neq q_2$, then q_1 and q_2 split P.

Let us observe that a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is special polygon if:

(Def.2) There exist p_1, p_2 such that p_1 and p_2 split it.

We introduce special polygonal as a synonym of special polygon.

Let us consider r_1, r_2, r'_1, r'_2. The functor $[.r_1, r_2, r'_1, r'_2.]$ yields a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by the condition (Def.3).

(Def.3) $[.r_1, r_2, r'_1, r'_2.] = \{p : p_{\mathbf{1}} = r_1 \wedge p_{\mathbf{2}} \leq r'_2 \wedge p_{\mathbf{2}} \geq r'_1 \vee p_{\mathbf{1}} \leq r_2 \wedge p_{\mathbf{1}} \geq r_1 \wedge p_{\mathbf{2}} = r'_2 \vee p_{\mathbf{1}} \leq r_2 \wedge p_{\mathbf{1}} \geq r_1 \wedge p_{\mathbf{2}} = r'_1 \vee p_{\mathbf{1}} = r_2 \wedge p_{\mathbf{2}} \leq r'_2 \wedge p_{\mathbf{2}} \geq r'_1\}$.

One can prove the following propositions:

(58) If $r_1 < r_2$ and $r'_1 < r'_2$, then $[.r_1, r_2, r'_1, r'_2.] = \mathcal{L}([r_1, r'_1], [r_1, r'_2]) \cup \mathcal{L}([r_1, r'_2], [r_2, r'_2]) \cup (\mathcal{L}([r_2, r'_2], [r_2, r'_1]) \cup \mathcal{L}([r_2, r'_1], [r_1, r'_1]))$.

(59) If $r_1 < r_2$ and $r'_1 < r'_2$, then $[.r_1, r_2, r'_1, r'_2.]$ is special polygonal.

(60) $\square_{\mathcal{E}^2} = [.0, 1, 0, 1.]$.

(61) $\square_{\mathcal{E}^2}$ is special polygonal.

One can verify the following observations:

* there exists a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ which is special polygonal,
* every subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ which is special polygonal is also non empty, and
* every subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ which is special polygonal is also non trivial.

A special polygon in $\mathbb{R}^2$ is a special polygonal subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$.

We now state four propositions:

(62) If P is then P is compact.

(63) Every special polygon in $\mathbb{R}^2$ is compact.

(64) If P is special polygonal, then for all p_1, p_2 such that $p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$ holds p_1 and p_2 split P.

(65) Suppose P is special polygonal. Given p_1, p_2. Suppose $p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$. Then there exist P_1, P_2 such that

(i) P_1 is a special polygonal arc joining p_1 and p_2,

(ii) P_2 is a special polygonal arc joining p_1 and p_2,

(iii) $P_1 \cap P_2 = \{p_1, p_2\}$, and

(iv) $P = P_1 \cup P_2$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. Some properties of restrictions of finite sequences. *Formalized Mathematics*, 5(**2**):241–245, 1996.

[7] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[8] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[9] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[10] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[11] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[12] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\rm T}$. Simple closed curves. *Formalized Mathematics*, 2(**5**):663–664, 1991.

[13] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[14] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.

[15] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Formalized Mathematics*, 3(**2**):275–278, 1992.

[16] Yatsuka Nakamura and Czesław Byliński. Extremal properties of vertices on special polygons, part I. *Formalized Mathematics*, 5(**1**):97–102, 1996.

[17] Yatsuka Nakamura and Jarosław Kotowicz. Connectedness conditions using polygonal arcs. *Formalized Mathematics*, 3(**1**):101–106, 1992.

[18] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[19] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[20] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.

[21] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[22] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[23] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.

[24] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[25] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[26] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[27] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received January 30, 1995