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On the Monoid of Endomorphisms of Universal Algebra and Many Sorted Algebra

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The articles [17], [7], [18], [5], [6], [4], [14], [16], [1], [12], [3], [10], [11], [8], [9], [2], [13], and [15] provide the terminology and notation for this paper.

In this paper U_1 is a universal algebra and f is a function from U_1 into U_1.

Let us consider U_1. The functor $\mathrm{end}(U_1)$ yields a non empty set of functions from the carrier of U_1 to the carrier of U_1 and is defined as follows:

(Def.1) For every function h from U_1 into U_1 holds $h \in \mathrm{end}(U_1)$ iff h is a homomorphism of U_1 into U_1.

Next we state four propositions:

(1) $\mathrm{end}(U_1) \subseteq (\text{the carrier of } U_1)^{\text{the carrier of } U_1}$.

(2) For every f holds $f \in \mathrm{end}(U_1)$ iff f is a homomorphism of U_1 into U_1.

(3) $\mathrm{id}_{(\text{the carrier of } U_1)} \in \mathrm{end}(U_1)$.

(4) For all elements f_1, f_2 of $\mathrm{end}(U_1)$ holds $f_1 \cdot f_2 \in \mathrm{end}(U_1)$.

Let us consider U_1. The functor $\mathrm{Comp}(U_1)$ yielding a binary operation on $\mathrm{end}(U_1)$ is defined as follows:

(Def.2) For all elements x, y of $\mathrm{end}(U_1)$ holds $(\mathrm{Comp}(U_1))(x, y) = y \cdot x$.

Let us consider U_1. The functor $\mathrm{End}(U_1)$ yields a strict multiplicative loop structure and is defined by:

(Def.3) The carrier of $\mathrm{End}(U_1) = \mathrm{end}(U_1)$ and the multiplication of $\mathrm{End}(U_1) = \mathrm{Comp}(U_1)$ and the unity of $\mathrm{End}(U_1) = \mathrm{id}_{(\text{the carrier of } U_1)}$.

Let us consider U_1. One can check that $\mathrm{End}(U_1)$ is non empty.

Let us consider U_1. One can verify that $\mathrm{End}(U_1)$ is left unital well unital and associative.

Next we state two propositions:

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(5) Let x, y be elements of the carrier of $\mathrm{End}(U_1)$ and let f, g be elements of $\mathrm{end}(U_1)$. If $x = f$ and $y = g$, then $x \cdot y = g \cdot f$.

(6) $\mathrm{id}_{(\text{the carrier of } U_1)} = \mathbf{1}_{\mathrm{End}(U_1)}$.

In the sequel S will be a non void non empty many sorted signature and U_2 will be a non-empty algebra over S.

Let us consider S, U_2. The functor $\mathrm{end}(U_2)$ yields a set of many sorted functions from the sorts of U_2 into the sorts of U_2 and is defined by the conditions (Def.4).

(Def.4) (i) Every element of $\mathrm{end}(U_2)$ is a many sorted function from U_2 into U_2, and

(ii) for every many sorted function h from U_2 into U_2 holds $h \in \mathrm{end}(U_2)$ iff h is a homomorphism of U_2 into U_2.

One can prove the following propositions:

(7) For every many sorted function F from U_2 into U_2 holds $F \in \mathrm{end}(U_2)$ iff F is a homomorphism of U_2 into U_2.

(8) For every element f of $\mathrm{end}(U_2)$ holds $f \in \prod \mathrm{MSFuncs}($the sorts of U_2, the sorts of $U_2)$.

(9) $\mathrm{end}(U_2) \subseteq \prod \mathrm{MSFuncs}($the sorts of U_2, the sorts of $U_2)$.

(10) $\mathrm{id}_{(\text{the sorts of } U_2)} \in \mathrm{end}(U_2)$.

(11) For all elements f_1, f_2 of $\mathrm{end}(U_2)$ holds $f_1 \circ f_2 \in \mathrm{end}(U_2)$.

(12) For every many sorted function F from $\mathrm{MSAlg}(U_1)$ into $\mathrm{MSAlg}(U_1)$ and for every element f of $\mathrm{end}(U_1)$ such that $F = \{0\} \longmapsto f$ holds $F \in \mathrm{end}(\mathrm{MSAlg}(U_1))$.

Let us consider S, U_2. The functor $\mathrm{Comp}(U_2)$ yielding a binary operation on $\mathrm{end}(U_2)$ is defined as follows:

(Def.5) For all elements x, y of $\mathrm{end}(U_2)$ holds $(\mathrm{Comp}(U_2))(x, y) = y \circ x$.

Let us consider S, U_2. The functor $\mathrm{End}(U_2)$ yields a strict multiplicative loop structure and is defined by:

(Def.6) The carrier of $\mathrm{End}(U_2) = \mathrm{end}(U_2)$ and the multiplication of $\mathrm{End}(U_2) = \mathrm{Comp}(U_2)$ and the unity of $\mathrm{End}(U_2) = \mathrm{id}_{(\text{the sorts of } U_2)}$.

Let us consider S, U_2. Note that $\mathrm{End}(U_2)$ is non empty.

Let us consider S, U_2. Note that $\mathrm{End}(U_2)$ is left unital well unital and associative.

The following four propositions are true:

(13) Let x, y be elements of the carrier of $\mathrm{End}(U_2)$ and let f, g be elements of $\mathrm{end}(U_2)$. If $x = f$ and $y = g$, then $x \cdot y = g \circ f$.

(14) $\mathrm{id}_{(\text{the sorts of } U_2)} = \mathbf{1}_{\mathrm{End}(U_2)}$.

(15) Let U_3, U_4 be universal algebras. Suppose U_3 and U_4 are similar. Let F be a many sorted function from $\mathrm{MSAlg}(U_3)$ into $(\mathrm{MSAlg}(U_4)$ over $\mathrm{MSSign}(U_3))$. Then $F(0)$ is a function from U_3 into U_4.

(16) For every element f of $\mathrm{end}(U_1)$ holds $\{0\} \longmapsto f$ is a many sorted function from $\mathrm{MSAlg}(U_1)$ into $\mathrm{MSAlg}(U_1)$.

Let G, H be multiplicative loop structures.

(Def.7) A function from the carrier of G into the carrier of H is called a map from G into H.

Let G, H be non empty multiplicative loop structures. A map from G into H is multiplicative if:

(Def.8) For all elements x, y of the carrier of G holds $\mathrm{it}(x \cdot y) = \mathrm{it}(x) \cdot \mathrm{it}(y)$.

A map from G into H is unity-preserving if:

(Def.9) $\mathrm{It}(1_G) = 1_H$.

Let us mention that there exists a non empty multiplicative loop structure which is left unital.

Let G, H be left unital non empty multiplicative loop structures. Observe that there exists a map from G into H which is multiplicative and unity-preserving.

Let G, H be left unital non empty multiplicative loop structures. A homomorphism from G to H is a multiplicative unity-preserving map from G into H.

Let G, H be left unital non empty multiplicative loop structures and let h be a map from G into H. We say that h is a monomorphism if and only if:

(Def.10) h is one-to-one.

We say that h is an epimorphism if and only if:

(Def.11) $\mathrm{rng}\, h =$ the carrier of H.

Let G, H be left unital non empty multiplicative loop structures and let h be a map from G into H. We say that h is an isomorphism if and only if:

(Def.12) h is an epimorphism and a monomorphism.

We now state the proposition

(17) Let G be a left unital non empty multiplicative loop structure. Then $\mathrm{id}_{(\text{the carrier of } G)}$ is a homomorphism from G to G.

Let G, H be left unital non empty multiplicative loop structures. We say that G and H are isomorphic if and only if:

(Def.13) There exists homomorphism from G to H which is an isomorphism.

Let us observe that this predicate is reflexive.

One can prove the following propositions:

(18) Let h be a function. Suppose $\mathrm{dom}\, h = \mathrm{end}(U_1)$ and for arbitrary x such that $x \in \mathrm{end}(U_1)$ holds $h(x) = \{0\} \longmapsto x$. Then h is a homomorphism from $\mathrm{End}(U_1)$ to $\mathrm{End}(\mathrm{MSAlg}(U_1))$.

(19) Let h be a homomorphism from $\mathrm{End}(U_1)$ to $\mathrm{End}(\mathrm{MSAlg}(U_1))$. Suppose that for arbitrary x such that $x \in \mathrm{end}(U_1)$ holds $h(x) = \{0\} \longmapsto x$. Then h is an isomorphism.

(20) $\mathrm{End}(U_1)$ and $\mathrm{End}(\mathrm{MSAlg}(U_1))$ are isomorphic.

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