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Associated Matrix of Linear Map

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The notation and terminology used in this paper are introduced in the following articles: [13], [2], [11], [17], [18], [33], [21], [32], [3], [34], [8], [9], [4], [14], [15], [35], [36], [23], [31], [16], [30], [26], [24], [12], [29], [19], [27], [1], [7], [25], [6], [10], [5], [22], [28], and [20].

1. Preliminaries

For simplicity we follow the rules: k, t, i, j, m, n are natural numbers, x is arbitrary, A is a set, and D is a non empty set.

We now state two propositions:

(1) For every finite sequence p of elements of D and for every i holds $p_{\restriction i}$ is a finite sequence of elements of D.

(2) For every i and for every finite sequence p holds $\operatorname{rng}(p_{\restriction i}) \subseteq \operatorname{rng} p$.

Let D be a non empty set. A matrix over D is a tabular finite sequence of elements of D^*.

Let K be a field. A matrix over K is a matrix over the carrier of K.

Let D be a non empty set, let us consider k, and let M be a matrix over D. Then $M_{\restriction k}$ is a matrix over D.

Next we state four propositions:

(3) For every finite sequence M of elements of D such that $\operatorname{len} M = n + 1$ holds $\operatorname{len}(M_{\restriction n+1}) = n$.

(4) Let M be a matrix over D of dimension $n + 1 \times m$ and let M_1 be a matrix over D. Then if $n > 0$, then $\operatorname{width} M = \operatorname{width}(M_{\restriction n+1})$ and if $M_1 = \langle M(n + 1)\rangle$, then $\operatorname{width} M = \operatorname{width} M_1$.

(5) For every matrix M over D of dimension $n + 1 \times m$ holds $M_{\restriction n+1}$ is a matrix over D of dimension $n \times m$.

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(6) For every finite sequence M of elements of D such that $\operatorname{len} M = n + 1$ holds $M = (M_{\restriction \operatorname{len} M}) \frown \langle M(\operatorname{len} M)\rangle$.

Let us consider D and let P be a finite sequence of elements of D. Then $\langle P\rangle$ is a matrix over D of dimension $1 \times \operatorname{len} P$.

2. More on Finite Sequence

One can prove the following propositions:

(7) For every set A and for every finite sequence F holds $(\mathrm{Sgm}(F^{-1} A)) \frown \mathrm{Sgm}(F^{-1} (\operatorname{rng} F \setminus A))$ is a permutation of $\operatorname{dom} F$.

(8) Let F be a finite sequence and let A be a subset of $\operatorname{rng} F$. Suppose F is one-to-one. Then there exists a permutation p of $\operatorname{dom} F$ such that $(F - A^{\mathrm{c}}) \frown (F - A) = F \cdot p$.

A function is finite sequence yielding if:

(Def.1) For every x such that $x \in \operatorname{dom}$ it holds $\mathrm{it}(x)$ is a finite sequence.

Let us observe that there exists a function which is finite sequence yielding.

Let F, G be finite sequence yielding functions. The functor $F \frown G$ yields a finite sequence yielding function and is defined by the conditions (Def.2).

(Def.2) (i) $\operatorname{dom}(F \frown G) = \operatorname{dom} F \cap \operatorname{dom} G$, and

(ii) for arbitrary i such that $i \in \operatorname{dom}(F \frown G)$ and for all finite sequences f, g such that $f = F(i)$ and $g = G(i)$ holds $(F \frown G)(i) = f \frown g$.

3. Matrices and Finite Sequences in Vector Space

For simplicity we adopt the following convention: K denotes a field, V denotes a vector space over K, a denotes an element of the carrier of K, W denotes an element of the carrier of V, K_1, K_2, K_3 denote linear combinations of V, and X denotes a subset of the carrier of V.

Next we state four propositions:

(9) If X is linearly independent and $\operatorname{support} K_1 \subseteq X$ and $\operatorname{support} K_2 \subseteq X$ and $\sum K_1 = \sum K_2$, then $K_1 = K_2$.

(10) If X is linearly independent and $\operatorname{support} K_1 \subseteq X$ and $\operatorname{support} K_2 \subseteq X$ and $\operatorname{support} K_3 \subseteq X$ and $\sum K_1 = \sum K_2 + \sum K_3$, then $K_1 = K_2 + K_3$.

(11) If X is linearly independent and $\operatorname{support} K_1 \subseteq X$ and $\operatorname{support} K_2 \subseteq X$ and $a \neq 0_K$ and $\sum K_1 = a \cdot \sum K_2$, then $K_1 = a \cdot K_2$.

(12) For every basis b_2 of V there exists a linear combination K_4 of V such that $W = \sum K_4$ and $\operatorname{support} K_4 \subseteq b_2$.

Let K be a field and let V be a vector space over K. We say that V is finite dimensional if and only if:

(Def.3) There exists finite subset of the carrier of V which is a basis of V.

Let K be a field. Note that there exists a vector space over K which is strict and finite dimensional.

Let K be a field and let V be a finite dimensional vector space over K. A finite sequence of elements of the carrier of V is called an ordered basis of V if:

(Def.4) It is one-to-one and rng it is a basis of V.

For simplicity we adopt the following convention: p will denote a finite sequence, M_1 will denote a matrix over D of dimension $n \times m$, M_2 will denote a matrix over D of dimension $k \times m$, V_1, V_2, V_3 will denote finite dimensional vector spaces over K, f, f_1, f_2 will denote maps from V_1 into V_2, g will denote a map from V_2 into V_3, b_1 will denote an ordered basis of V_1, b_2 will denote an ordered basis of V_2, b_3 will denote an ordered basis of V_3, b will denote a basis of V_1, v_1, v_2 will denote vectors of V_2, v will denote an element of the carrier of V_1, p_2, F will denote finite sequences of elements of the carrier of V_1, p_1, d will denote finite sequences of elements of the carrier of K, and K_4 will denote a linear combination of V_1.

Let us consider K, let us consider V_1, V_2, and let us consider f_1, f_2. The functor $f_1 + f_2$ yielding a map from V_1 into V_2 is defined as follows:

(Def.5) For every element v of the carrier of V_1 holds $(f_1+f_2)(v) = f_1(v)+f_2(v)$.

Let us consider K, let us consider V_1, V_2, let us consider f, and let a be an element of the carrier of K. The functor $a \cdot f$ yielding a map from V_1 into V_2 is defined as follows:

(Def.6) For every element v of the carrier of V_1 holds $(a \cdot f)(v) = a \cdot f(v)$.

The following propositions are true:

(13) Let a be an element of the carrier of V_1, and let F be a finite sequence of elements of the carrier of V_1, and let G be a finite sequence of elements of the carrier of K. Suppose $\mathrm{len}\, F = \mathrm{len}\, G$ and for every k and for every element v of the carrier of K such that $k \in \mathrm{dom}\, F$ and $v = G(k)$ holds $F(k) = v \cdot a$. Then $\sum F = \sum G \cdot a$.

(14) Let a be an element of the carrier of V_1, and let F be a finite sequence of elements of the carrier of K, and let G be a finite sequence of elements of the carrier of V_1. If $\mathrm{len}\, F = \mathrm{len}\, G$ and for every k such that $k \in \mathrm{dom}\, F$ holds $G(k) = \pi_k F \cdot a$, then $\sum G = \sum F \cdot a$.

(15) If for every k such that $k \in \mathrm{dom}\, F$ holds $\pi_k F = 0_{(V_1)}$, then $\sum F = 0_{(V_1)}$.

Let us consider K, let us consider V_1, and let us consider p_1, p_2. The functor $\mathrm{lmlt}(p_1, p_2)$ yielding a finite sequence of elements of the carrier of V_1 is defined as follows:

(Def.7) $\mathrm{lmlt}(p_1, p_2) = (\text{the left multiplication of } V_1)^{\circ}(p_1, p_2)$.

Next we state the proposition

(16) If $\mathrm{dom}\, p_1 = \mathrm{dom}\, p_2$, then $\mathrm{dom}\,\mathrm{lmlt}(p_1, p_2) = \mathrm{dom}\, p_1$ and $\mathrm{dom}\,\mathrm{lmlt}(p_1, p_2) = \mathrm{dom}\, p_2$.

Let us consider K, let us consider V_1, and let M be a matrix over the carrier of V_1. The functor $\sum M$ yields a finite sequence of elements of the carrier of V_1 and is defined as follows:

(Def.8) $\operatorname{len}\sum M = \operatorname{len} M$ and for every k such that $k \in \operatorname{dom}\sum M$ holds $\pi_k \sum M = \sum \operatorname{Line}(M, k)$.

The following propositions are true:

(17) For every matrix M over the carrier of V_1 such that $\operatorname{len} M = 0$ holds $\sum\sum M = 0_{(V_1)}$.

(18) For every matrix M over the carrier of V_1 of dimension $m+1 \times 0$ holds $\sum\sum M = 0_{(V_1)}$.

(19) For every element x of the carrier of V_1 holds $\langle\langle x\rangle\rangle = \langle\langle x\rangle\rangle^{\mathrm{T}}$.

(20) For every finite sequence p of elements of the carrier of V_1 such that f is linear holds $f(\sum p) = \sum(f \cdot p)$.

(21) Let a be a finite sequence of elements of the carrier of K and let p be a finite sequence of elements of the carrier of V_1. If $\operatorname{len} p = \operatorname{len} a$, then if f is linear, then $f \cdot \operatorname{lmlt}(a, p) = \operatorname{lmlt}(a, f \cdot p)$.

(22) Let a be a finite sequence of elements of the carrier of K. If $\operatorname{len} a = \operatorname{len} b_2$, then if g is linear, then $g(\sum \operatorname{lmlt}(a, b_2)) = \sum \operatorname{lmlt}(a, g \cdot b_2)$.

(23) Let F, F_1 be finite sequences of elements of the carrier of V_1, and let K_4 be a linear combination of V_1, and let p be a permutation of $\operatorname{dom} F$. If $F_1 = F \cdot p$, then $K_4\, F_1 = (K_4\, F) \cdot p$.

(24) If F is one-to-one and support $K_4 \subseteq \operatorname{rng} F$, then $\sum(K_4\, F) = \sum K_4$.

(25) Let A be a set and let p be a finite sequence of elements of the carrier of V_1. Suppose $\operatorname{rng} p \subseteq A$. Suppose f_1 is linear and f_2 is linear and for every v such that $v \in A$ holds $f_1(v) = f_2(v)$. Then $f_1(\sum p) = f_2(\sum p)$.

(26) If f_1 is linear and f_2 is linear, then for every ordered basis b_1 of V_1 such that $\operatorname{len} b_1 > 0$ holds if $f_1 \cdot b_1 = f_2 \cdot b_1$, then $f_1 = f_2$.

Let D be a non empty set. Observe that every matrix over D is finite sequence yielding.

Let D be a non empty set and let F, G be matrices over D. Then $F \frown G$ is a matrix over D.

Let D be a non empty set, let us consider n, m, k, let M_1 be a matrix over D of dimension $n \times k$, and let M_2 be a matrix over D of dimension $m \times k$. Then $M_1 \frown M_2$ is a matrix over D of dimension $n + m \times k$.

One can prove the following propositions:

(27) Given i, and let M_1 be a matrix over D of dimension $n \times k$, and let M_2 be a matrix over D of dimension $m \times k$. If $i \in \operatorname{dom} M_1$, then $\operatorname{Line}(M_1 \frown M_2, i) = \operatorname{Line}(M_1, i)$.

(28) Let M_1 be a matrix over D of dimension $n \times k$ and let M_2 be a matrix over D of dimension $m \times k$. If $\operatorname{width} M_1 = \operatorname{width} M_2$, then $\operatorname{width}(M_1 \frown M_2) = \operatorname{width} M_1$ and $\operatorname{width}(M_1 \frown M_2) = \operatorname{width} M_2$.

(29) Given i, n, and let M_1 be a matrix over D of dimension $t \times k$, and let M_2 be a matrix over D of dimension $m \times k$. If $n \in \operatorname{dom} M_2$ and $i = \operatorname{len} M_1 + n$, then $\operatorname{Line}(M_1 \frown M_2, i) = \operatorname{Line}(M_2, n)$.

(30) Let M_1 be a matrix over D of dimension $n \times k$ and let M_2 be a matrix over D of dimension $m \times k$. If $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$, then for every i such that $i \in \mathrm{Seg}\, \mathrm{width}\, M_1$ holds $(M_1 \frown M_2)_{\square,i} = ((M_1)_{\square,i}) \frown ((M_2)_{\square,i})$.

(31) Let M_1 be a matrix over the carrier of V_1 of dimension $n \times k$ and let M_2 be a matrix over the carrier of V_1 of dimension $m \times k$. Then $\sum(M_1 \frown M_2) = (\sum M_1) \frown \sum M_2$.

(32) Let M_1 be a matrix over D of dimension $n \times k$ and let M_2 be a matrix over D of dimension $m \times k$. If $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$, then $(M_1 \frown M_2)^{\mathrm{T}} = ({M_1}^{\mathrm{T}}) \frown {M_2}^{\mathrm{T}}$.

(33) For all matrices M_1, M_2 over the carrier of V_1 holds (the addition of $V_1)^{\circ}(\sum M_1, \sum M_2) = \sum(M_1 \frown M_2)$.

Let D be a non empty set, let F be a binary operation on D, and let P_1, P_2 be finite sequences of elements of D. Then $F^{\circ}(P_1, P_2)$ is a finite sequence of elements of D.

Next we state several propositions:

(34) Let P_1, P_2 be finite sequences of elements of the carrier of V_1. If $\mathrm{len}\, P_1 = \mathrm{len}\, P_2$, then $\sum$((the addition of $V_1)^{\circ}(P_1, P_2)) = \sum P_1 + \sum P_2$.

(35) For all matrices M_1, M_2 over the carrier of V_1 such that $\mathrm{len}\, M_1 = \mathrm{len}\, M_2$ holds $\sum\sum M_1 + \sum\sum M_2 = \sum\sum(M_1 \frown M_2)$.

(36) For every finite sequence P of elements of the carrier of V_1 holds $\sum\sum\langle P\rangle = \sum\sum(\langle P\rangle^{\mathrm{T}})$.

(37) For every n and for every matrix M over the carrier of V_1 such that $\mathrm{len}\, M = n$ holds $\sum\sum M = \sum\sum(M^{\mathrm{T}})$.

(38) Let M be a matrix over the carrier of K of dimension $n \times m$. Suppose $n > 0$ and $m > 0$. Let p, d be finite sequences of elements of the carrier of K. Suppose $\mathrm{len}\, p = n$ and $\mathrm{len}\, d = m$ and for every j such that $j \in \mathrm{dom}\, d$ holds $\pi_j d = \sum(p \bullet M_{\square,j})$. Let b, c be finite sequences of elements of the carrier of V_1. Suppose $\mathrm{len}\, b = m$ and $\mathrm{len}\, c = n$ and for every i such that $i \in \mathrm{dom}\, c$ holds $\pi_i c = \sum \mathrm{lmlt}(\mathrm{Line}(M,i), b)$. Then $\sum \mathrm{lmlt}(p,c) = \sum \mathrm{lmlt}(d,b)$.

4. Decomposition of a Vector in Basis

Let K be a field, let V be a finite dimensional vector space over K, let b_1 be an ordered basis of V, and let W be an element of the carrier of V. The functor $W \rightarrow b_1$ yielding a finite sequence of elements of the carrier of K is defined by the conditions (Def.9).

(Def.9) (i) $\mathrm{len}(W \rightarrow b_1) = \mathrm{len}\, b_1$, and

(ii) there exists a linear combination K_4 of V such that $W = \sum K_4$ and $\mathrm{support}\, K_4 \subseteq \mathrm{rng}\, b_1$ and for every k such that $1 \leq k$ and $k \leq \mathrm{len}(W \rightarrow b_1)$ holds $\pi_k(W \rightarrow b_1) = K_4(\pi_k b_1)$.

The following four propositions are true:

(39) If $v_1 \to b_2 = v_2 \to b_2$, then $v_1 = v_2$.

(40) $v = \sum \mathrm{lmlt}(v \to b_1, b_1)$.

(41) If $\mathrm{len}\, d = \mathrm{len}\, b_1$, then $d = \sum \mathrm{lmlt}(d, b_1) \to b_1$.

(42) Let a be a finite sequence of elements of the carrier of K. Suppose $\mathrm{len}\, a = \mathrm{len}\, b_2$. Let j be a natural number. Suppose $j \in \mathrm{dom}\, b_3$. Let d be a finite sequence of elements of the carrier of K. Suppose $\mathrm{len}\, d = \mathrm{len}\, b_2$ and for every k such that $k \in \mathrm{dom}\, b_2$ holds $d(k) = \pi_j(g(\pi_k b_2) \to b_3)$. If $\mathrm{len}\, b_2 > 0$ and $\mathrm{len}\, b_3 > 0$, then $\pi_j(\sum \mathrm{lmlt}(a, g \cdot b_2) \to b_3) = \sum(a \bullet d)$.

5. Associated Matrix of Linear Map

Let K be a field, let V_1, V_2 be finite dimensional vector spaces over K, let f be a function from the carrier of V_1 into the carrier of V_2, let b_1 be a finite sequence of elements of the carrier of V_1, and let b_2 be an ordered basis of V_2. The functor $\mathrm{AutMt}(f, b_1, b_2)$ yielding a matrix over K is defined as follows:

(Def.10) $\mathrm{len}\, \mathrm{AutMt}(f, b_1, b_2) = \mathrm{len}\, b_1$ and for every k such that $k \in \mathrm{dom}\, b_1$ holds $\pi_k\, \mathrm{AutMt}(f, b_1, b_2) = f(\pi_k b_1) \to b_2$.

One can prove the following propositions:

(43) If $\mathrm{len}\, b_1 = 0$, then $\mathrm{AutMt}(f, b_1, b_2) = \varepsilon$.

(44) If $\mathrm{len}\, b_1 > 0$, then $\mathrm{width}\, \mathrm{AutMt}(f, b_1, b_2) = \mathrm{len}\, b_2$.

(45) If f_1 is linear and f_2 is linear, then if $\mathrm{AutMt}(f_1, b_1, b_2) = \mathrm{AutMt}(f_2, b_1, b_2)$ and $\mathrm{len}\, b_1 > 0$, then $f_1 = f_2$.

(46) If f is linear and g is linear and $\mathrm{len}\, b_1 > 0$ and $\mathrm{len}\, b_2 > 0$ and $\mathrm{len}\, b_3 > 0$, then $\mathrm{AutMt}(g \cdot f, b_1, b_3) = \mathrm{AutMt}(f, b_1, b_2) \cdot \mathrm{AutMt}(g, b_2, b_3)$.

(47) $\mathrm{AutMt}(f_1 + f_2, b_1, b_2) = \mathrm{AutMt}(f_1, b_1, b_2) + \mathrm{AutMt}(f_2, b_1, b_2)$.

(48) If $a \neq 0_K$, then $\mathrm{AutMt}(a \cdot f, b_1, b_2) = a \cdot \mathrm{AutMt}(f, b_1, b_2)$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[5] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[6] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.

[7] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[10] Czesław Byliński. Semigroup operations on finite subsets. *Formalized Mathematics*, 1(**4**):651–656, 1990.

[11] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[12] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[13] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[14] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.

[15] Katarzyna Jankowska. Transpose matrices and groups of permutations. *Formalized Mathematics*, 2(**5**):711–717, 1991.

[16] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[17] Michał Muzalewski. Categories of groups. *Formalized Mathematics*, 2(**4**):563–571, 1991.

[18] Michał Muzalewski. Rings and modules - part II. *Formalized Mathematics*, 2(**4**):579–585, 1991.

[19] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[20] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(**2**):369–376, 1990.

[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[22] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[23] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(**1**):187–190, 1990.

[24] Wojciech A. Trybulec. Basis of vector space. *Formalized Mathematics*, 1(**5**):883–885, 1990.

[25] Wojciech A. Trybulec. Linear combinations in real linear space. *Formalized Mathematics*, 1(**3**):581–588, 1990.

[26] Wojciech A. Trybulec. Linear combinations in vector space. *Formalized Mathematics*, 1(**5**):877–882, 1990.

[27] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.

[28] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.

[29] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[30] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Formalized Mathematics*, 1(**5**):865–870, 1990.

[31] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[32] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[33] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[34] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[35] Katarzyna Zawadzka. The product and the determinant of matrices with entries in a field. *Formalized Mathematics*, 4(**1**):1–8, 1993.

[36] Katarzyna Zawadzka. The sum and product of finite sequences of elements of a field. *Formalized Mathematics*, 3(**2**):205–211, 1992.

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