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On the Lattice of Subalgebras of a Universal Algebra

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The papers [13], [16], [15], [12], [17], [6], [7], [2], [8], [5], [4], [18], [1], [10], [11], [9], [14], and [3] provide the terminology and notation for this paper.

In this paper U_0 is a universal algebra, H is a non empty subset of the carrier of U_0, and o is an operation of U_0.

Let us consider U_0. Family of subalgebras of U_0 is defined by:

(Def.1) For arbitrary U_1 such that $U_1 \in$ it holds U_1 is a subalgebra of U_0.

Let us consider U_0. One can check that there exists a family of subalgebras of U_0 which is non empty.

Let us consider U_0. Then Subalgebras(U_0) is a non empty family of subalgebras of U_0. Let U_2 be a non empty family of subalgebras of U_0. We see that the element of U_2 is a subalgebra of U_0.

Let us consider U_0. Then $\bigsqcup_{(U_0)}$ is a binary operation on Subalgebras(U_0). Then $\bigsqcap_{(U_0)}$ is a binary operation on Subalgebras(U_0).

Let us consider U_0 and let u be an element of Subalgebras(U_0). The functor $\overline{u}$ yielding a subset of the carrier of U_0 is defined as follows:

(Def.2) There exists a subalgebra U_1 of U_0 such that $u = U_1$ and $\overline{u} =$ the carrier of U_1.

Let us consider U_0. The functor Carr(U_0) yields a function from Subalgebras(U_0) into $2^{\text{the carrier of } U_0}$ and is defined by:

(Def.3) For every element u of Subalgebras(U_0) holds $(\text{Carr}(U_0))(u) = \overline{u}$.

We now state several propositions:

(1) For arbitrary u holds $u \in$ Subalgebras(U_0) iff there exists a strict subalgebra U_1 of U_0 such that $u = U_1$.

(2) Let H be a non empty subset of the carrier of U_0 and given o. If arity $o = 0$, then H is closed on o iff $o(\varepsilon) \in H$.

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(3) For every subalgebra U_1 of U_0 holds the carrier of $U_1 \subseteq$ the carrier of U_0.

(4) For every non empty subset H of the carrier of U_0 and for every o such that H is closed on o and arity $o = 0$ holds $o_H = o$.

(5) If U_0 has constants, then Constants$(U_0) = \{o(\varepsilon) : o$ ranges over operation of U_0, arity $o = 0\}$.

(6) For every universal algebra U_0 with constants and for every subalgebra U_1 of U_0 holds Constants$(U_0) =$ Constants(U_1).

Let U_0 be a universal algebra with constants. Note that every subalgebra of U_0 has constants.

The following proposition is true

(7) For every universal algebra U_0 with constants and for all subalgebras U_1, U_3 of U_0 holds Constants$(U_1) =$ Constants(U_3).

Let us consider U_0. Then Carr(U_0) is a function from Subalgebras(U_0) into $2^{\text{the carrier of } U_0}$ and it can be characterized by the condition:

(Def.4) For every element u of Subalgebras(U_0) and for every subalgebra U_1 of U_0 such that $u = U_1$ holds $(\text{Carr}(U_0))(u) =$ the carrier of U_1.

One can prove the following propositions:

(8) For every strict subalgebra H of U_0 and for every element u of U_0 holds $u \in (\text{Carr}(U_0))(H)$ iff $u \in H$.

(9) For every non empty subset H of Subalgebras(U_0) holds $(\text{Carr}(U_0))^{\circ} H$ is non empty.

(10) For every universal algebra U_0 with constants and for every strict subalgebra U_1 of U_0 holds Constants$(U_0) \subseteq (\text{Carr}(U_0))(U_1)$.

(11) Let U_0 be a universal algebra with constants, and let U_1 be a subalgebra of U_0, and let a be arbitrary. If a is an element of Constants(U_0), then $a \in$ the carrier of U_1.

(12) Let U_0 be a universal algebra with constants and let H be a non empty subset of Subalgebras(U_0). Then $\bigcap((\text{Carr}(U_0))^{\circ} H)$ is a non empty subset of the carrier of U_0.

(13) For every universal algebra U_0 with constants holds the carrier of the lattice of subalgebras of $U_0 =$ Subalgebras(U_0).

(14) Let U_0 be a universal algebra with constants, and let H be a non empty subset of Subalgebras(U_0), and let S be a non empty subset of the carrier of U_0. If $S = \bigcap((\text{Carr}(U_0))^{\circ} H)$, then S is operations closed.

Let U_0 be a strict universal algebra with constants and let H be a non empty subset of Subalgebras(U_0). The functor $\bigcap H$ yielding a strict subalgebra of U_0 is defined as follows:

(Def.5) The carrier of $\bigcap H = \bigcap((\text{Carr}(U_0))^{\circ} H)$.

One can prove the following propositions:

(15) Let U_0 be a universal algebra with constants, and let l_1, l_2 be elements of the carrier of the lattice of subalgebras of U_0, and let U_1, U_3 be strict subalgebras of U_0. If $l_1 = U_1$ and $l_2 = U_3$, then $l_1 \sqcup l_2 = U_1 \bigsqcup U_3$.

(16) Let U_0 be a universal algebra with constants, and let l_1, l_2 be elements of the carrier of the lattice of subalgebras of U_0, and let U_1, U_3 be strict subalgebras of U_0. If $l_1 = U_1$ and $l_2 = U_3$, then $l_1 \sqcap l_2 = U_1 \cap U_3$.

(17) Let U_0 be a universal algebra with constants, and let l_1, l_2 be elements of the carrier of the lattice of subalgebras of U_0, and let U_1, U_3 be strict subalgebras of U_0. Suppose $l_1 = U_1$ and $l_2 = U_3$. Then $l_1 \sqsubseteq l_2$ if and only if the carrier of $U_1 \subseteq$ the carrier of U_3.

(18) Let U_0 be a universal algebra with constants, and let l_1, l_2 be elements of the carrier of the lattice of subalgebras of U_0, and let U_1, U_3 be strict subalgebras of U_0. If $l_1 = U_1$ and $l_2 = U_3$, then $l_1 \sqsubseteq l_2$ iff U_1 is a subalgebra of U_3.

(19) For every strict universal algebra U_0 with constants holds the lattice of subalgebras of U_0 is bounded.

(20) For every strict universal algebra U_0 with constants and for every strict subalgebra U_1 of U_0 holds $\mathrm{Gen}^{\mathrm{UA}}(\mathrm{Constants}(U_0)) \cap U_1 = \mathrm{Gen}^{\mathrm{UA}}(\mathrm{Constants}(U_0))$.

(21) For every strict universal algebra U_0 with constants holds $\perp_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}^{\mathrm{UA}}(\mathrm{Constants}(U_0))$.

(22) Let U_0 be a strict universal algebra with constants, and let U_1 be a subalgebra of U_0, and let H be a subset of the carrier of U_0. If $H =$ the carrier of U_0, then $\mathrm{Gen}^{\mathrm{UA}}(H) \bigsqcup U_1 = \mathrm{Gen}^{\mathrm{UA}}(H)$.

(23) Let U_0 be a strict universal algebra with constants and let H be a subset of the carrier of U_0. Suppose $H =$ the carrier of U_0. Then $\top_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}^{\mathrm{UA}}(H)$.

(24) For every strict universal algebra U_0 with constants holds $\top_{\text{the lattice of subalgebras of } U_0} = U_0$.

(25) For every strict universal algebra U_0 with constants holds the lattice of subalgebras of U_0 is complete.

Let U_4, U_5 be universal algebras with constants and let F be a function from the carrier of U_4 into the carrier of U_5. The functor $\mathrm{FuncLatt}(F)$ yielding a function from the carrier of the lattice of subalgebras of U_4 into the carrier of the lattice of subalgebras of U_5 is defined by the condition (Def.6).

(Def.6) Let U_1 be a strict subalgebra of U_4 and let H be a subset of the carrier of U_5. If $H = F^{\circ}$(the carrier of U_1), then $(\mathrm{FuncLatt}(F))(U_1) = \mathrm{Gen}^{\mathrm{UA}}(H)$.

We now state the proposition

(26) Let U_0 be a strict universal algebra with constants and let F be a function from the carrier of U_0 into the carrier of U_0. Suppose $F = \mathrm{id}_{(\text{the carrier of } U_0)}$. Then $\mathrm{FuncLatt}(F) = \mathrm{id}_{(\text{the carrier of the lattice of subalgebras of } U_0)}$.

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