FORMALIZED MATHEMATICS
Volume 5, Number 3, 1996
Warsaw University - Białystok

The Theorem of Weierstrass

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Summary. The basic purpose of this article is to prove the important Weierstrass' theorem which states that a real valued continuous function f on a topological space T assumes a maximum and a minimum value on the compact subset S of T, i.e., there exist points x_1, x_2 of T being elements of S, such that $f(x_1)$ and $f(x_2)$ are the supremum and the infimum, respectively, of $f(S)$, which is the image of S under the function f. The paper is divided into three parts. In the first part, we prove some auxiliary theorems concerning properties of balls in metric spaces and define special families of subsets of topological spaces. These concepts are used in the next part of the paper which contains the essential part of the article, namely the formalization of the proof of Weierstrass' theorem. Here, we also prove a theorem concerning the compactness of images of compact sets of T under a continuous function. The final part of this work is developed for the purpose of defining some measures of the distance between compact subsets of topological metric spaces. Some simple theorems about these measures are also proved.

MML Identifier: `WEIERSTR`.

The papers [31], [36], [9], [32], [30], [35], [29], [37], [7], [8], [5], [6], [27], [2], [15], [1], [14], [17], [10], [21], [19], [20], [18], [25], [33], [34], [3], [13], [22], [24], [38], [12], [26], [11], [4], [23], [28], and [16] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) Let M be a metric space, and let x_1, x_2 be points of M, and let r_1, r_2 be real numbers. Then there exists a point x of M and there exists a real number r such that $\mathrm{Ball}(x_1, r_1) \cup \mathrm{Ball}(x_2, r_2) \subseteq \mathrm{Ball}(x, r)$.

ISSN 1426–2630

(2) Let M be a metric space, and let n be a natural number, and let F be a family of subsets of M, and let p be a finite sequence. Suppose F is finite and a family of balls and $\operatorname{rng} p = F$ and $\operatorname{dom} p = \operatorname{Seg}(n+1)$. Then there exists a family G of subsets of M such that
(i) G is finite and a family of balls, and
(ii) there exists a finite sequence q such that $\operatorname{rng} q = G$ and $\operatorname{dom} q = \operatorname{Seg} n$ and there exists a point x of M and there exists a real number r such that $\bigcup F \subseteq \bigcup G \cup \operatorname{Ball}(x, r)$.

(3) Let M be a metric space and let F be a family of subsets of M. Suppose F is finite and a family of balls. Then there exists a point x of M and there exists a real number r such that $\bigcup F \subseteq \operatorname{Ball}(x, r)$.

Let T, S be topological spaces, let f be a map from T into S, and let G be a family of subsets of S. The functor $f^{-1}\, G$ yields a family of subsets of T and is defined by the condition (Def.1).

(Def.1) Let A be a subset of the carrier of T. Then $A \in f^{-1}\, G$ if and only if there exists a subset B of the carrier of S such that $B \in G$ and $A = f^{-1} B$.

Next we state two propositions:

(4) Let T, S be topological spaces, and let f be a map from T into S, and let A, B be families of subsets of S. If $A \subseteq B$, then $f^{-1}\, A \subseteq f^{-1}\, B$.

(5) Let T, S be topological spaces, and let f be a map from T into S, and let B be a family of subsets of S. If f is continuous and B is open, then $f^{-1}\, B$ is open.

Let T, S be topological spaces, let f be a map from T into S, and let G be a family of subsets of T. The functor $f^{\circ} G$ yields a family of subsets of S and is defined by the condition (Def.2).

(Def.2) Let A be a subset of the carrier of S. Then $A \in f^{\circ} G$ if and only if there exists a subset B of the carrier of T such that $B \in G$ and $A = f^{\circ} B$.

One can prove the following propositions:

(6) Let T, S be topological spaces, and let f be a map from T into S, and let A, B be families of subsets of T. If $A \subseteq B$, then $f^{\circ} A \subseteq f^{\circ} B$.

(7) Let T, S be topological spaces, and let f be a map from T into S, and let B be a family of subsets of S, and let P be a subset of the carrier of S. If $f^{\circ} f^{-1}\, B$ is a cover of P, then B is a cover of P.

(8) Let T, S be topological spaces, and let f be a map from T into S, and let B be a family of subsets of T, and let P be a subset of the carrier of T. If B is a cover of P, then $f^{-1}\, f^{\circ} B$ is a cover of P.

(9) Let T, S be topological spaces, and let f be a map from T into S, and let Q be a family of subsets of S. Then $\bigcup (f^{\circ} f^{-1}\, Q) \subseteq \bigcup Q$.

(10) Let T, S be topological spaces, and let f be a map from T into S, and let P be a family of subsets of T. Then $\bigcup P \subseteq \bigcup (f^{-1}\, f^{\circ} P)$.

(11) Let T, S be topological spaces, and let f be a map from T into S, and let Q be a family of subsets of S. If Q is finite, then $f^{-1}\, Q$ is finite.

(12) Let T, S be topological spaces, and let f be a map from T into S, and let P be a family of subsets of T. If P is finite, then $f^\circ P$ is finite.

(13) Let T, S be topological spaces, and let f be a map from T into S, and let P be a subset of the carrier of T, and let F be a family of subsets of S. Given a family B of subsets of T such that $B \subseteq f^{-1} F$ and B is a cover of P and finite. Then there exists a family G of subsets of S such that $G \subseteq F$ and G is a cover of $f^\circ P$ and finite.

2. The Weierstrass' Theorem

One can prove the following three propositions:

(14) Let T, S be topological spaces, and let f be a map from T into S, and let P be a subset of the carrier of T. If P is compact and f is continuous, then $f^\circ P$ is compact.

(15) Let T be a topological space, and let f be a map from T into $\mathbb{R}^{\mathbf{1}}$, and let P be a subset of the carrier of T. If P is compact and f is continuous, then $f^\circ P$ is compact.

(16) Let f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ and let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. If P is compact and f is continuous, then $f^\circ P$ is compact.

Let P be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$. The functor Ω_P yields a subset of $\mathbb{R}$ and is defined as follows:

(Def.3) $\Omega_P = P$.

Next we state three propositions:

(17) For every subset P of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that P is compact holds Ω_P is bounded.

(18) For every subset P of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that P is compact holds Ω_P is closed.

(19) For every subset P of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that P is compact holds Ω_P is compact.

Let P be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$. The functor $\sup P$ yields a real number and is defined as follows:

(Def.4) $\sup P = \sup(\Omega_P)$.

The functor $\inf P$ yielding a real number is defined by:

(Def.5) $\inf P = \inf(\Omega_P)$.

We now state two propositions:

(20) Let T be a topological space, and let f be a map from T into $\mathbb{R}^{\mathbf{1}}$, and let P be a subset of the carrier of T. Suppose $P \neq \emptyset$ and P is compact and f is continuous. Then there exists a point x_1 of T such that $x_1 \in P$ and $f(x_1) = \sup(f^\circ P)$.

(21) Let T be a topological space, and let f be a map from T into $\mathbb{R}^{\mathbf{1}}$, and let P be a subset of the carrier of T. Suppose $P \neq \emptyset$ and P is compact and f is continuous. Then there exists a point x_2 of T such that $x_2 \in P$ and $f(x_2) = \inf(f^\circ P)$.

3. The Measure of the Distance Between Compact Sets

Let M be a metric space and let x be a point of M. The functor $\operatorname{dist}(x)$ yielding a map from $M_{\rm top}$ into $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def.6) For every point y of M holds $(\operatorname{dist}(x))(y) = \rho(y, x)$.

The following three propositions are true:

(22) For every metric space M and for every point x of M holds $\operatorname{dist}(x)$ is continuous.

(23) Let M be a metric space, and let x be a point of M, and let P be a subset of the carrier of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact. Then there exists a point x_1 of $M_{\rm top}$ such that $x_1 \in P$ and $(\operatorname{dist}(x))(x_1) = \sup((\operatorname{dist}(x))^\circ P)$.

(24) Let M be a metric space, and let x be a point of M, and let P be a subset of the carrier of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact. Then there exists a point x_2 of $M_{\rm top}$ such that $x_2 \in P$ and $(\operatorname{dist}(x))(x_2) = \inf((\operatorname{dist}(x))^\circ P)$.

Let M be a metric space and let P be a subset of the carrier of $M_{\rm top}$. Let us assume that $P \neq \emptyset$ and P is compact. The functor $\operatorname{dist}_{\max}(P)$ yielding a map from $M_{\rm top}$ into $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def.7) For every point x of M holds $(\operatorname{dist}_{\max}(P))(x) = \sup((\operatorname{dist}(x))^\circ P)$.

The functor $\operatorname{dist}_{\min}(P)$ yields a map from $M_{\rm top}$ into $\mathbb{R}^{\mathbf{1}}$ and is defined by:

(Def.8) For every point x of M holds $(\operatorname{dist}_{\min}(P))(x) = \inf((\operatorname{dist}(x))^\circ P)$.

One can prove the following propositions:

(25) Let M be a metric space and let P be a subset of the carrier of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M. If $p_1 \in P$, then $\rho(p_1, p_2) \leq \sup((\operatorname{dist}(p_2))^\circ P)$ and $\inf((\operatorname{dist}(p_2))^\circ P) \leq \rho(p_1, p_2)$.

(26) Let M be a metric space and let P be a subset of the carrier of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M. Then $|\sup((\operatorname{dist}(p_1))^\circ P) - \sup((\operatorname{dist}(p_2))^\circ P)| \leq \rho(p_1, p_2)$.

(27) Let M be a metric space and let P be a subset of the carrier of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M and let x_1, x_2 be real numbers. If $x_1 = (\operatorname{dist}_{\max}(P))(p_1)$ and $x_2 = (\operatorname{dist}_{\max}(P))(p_2)$, then $|x_1 - x_2| \leq \rho(p_1, p_2)$.

(28) Let M be a metric space and let P be a subset of the carrier of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M. Then $|\inf((\operatorname{dist}(p_1))^\circ P) - \inf((\operatorname{dist}(p_2))^\circ P)| \leq \rho(p_1, p_2)$.

(29) Let M be a metric space and let P be a subset of the carrier of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M and let x_1,

x_2 be real numbers. If $x_1 = (\mathrm{dist}_{\mathrm{min}}(P))(p_1)$ and $x_2 = (\mathrm{dist}_{\mathrm{min}}(P))(p_2)$, then $|x_1 - x_2| \leq \rho(p_1, p_2)$.

(30) Let M be a metric space and let X be a subset of the carrier of M_{top}. If $X \neq \emptyset$ and X is compact, then $\mathrm{dist}_{\mathrm{max}}(X)$ is continuous.

(31) Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exists a point x_1 of M_{top} such that $x_1 \in Q$ and $(\mathrm{dist}_{\mathrm{max}}(P))(x_1) = \sup((\mathrm{dist}_{\mathrm{max}}(P))^{\circ}Q)$.

(32) Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exists a point x_2 of M_{top} such that $x_2 \in Q$ and $(\mathrm{dist}_{\mathrm{max}}(P))(x_2) = \inf((\mathrm{dist}_{\mathrm{max}}(P))^{\circ}Q)$.

(33) Let M be a metric space and let X be a subset of the carrier of M_{top}. If $X \neq \emptyset$ and X is compact, then $\mathrm{dist}_{\mathrm{min}}(X)$ is continuous.

(34) Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exists a point x_1 of M_{top} such that $x_1 \in Q$ and $(\mathrm{dist}_{\mathrm{min}}(P))(x_1) = \sup((\mathrm{dist}_{\mathrm{min}}(P))^{\circ}Q)$.

(35) Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exists a point x_2 of M_{top} such that $x_2 \in Q$ and $(\mathrm{dist}_{\mathrm{min}}(P))(x_2) = \inf((\mathrm{dist}_{\mathrm{min}}(P))^{\circ}Q)$.

Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Let us assume that $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. The functor $\mathrm{dist}^{\mathrm{min}}_{\mathrm{min}}(P, Q)$ yields a real number and is defined as follows:

(Def.9) $\mathrm{dist}^{\mathrm{min}}_{\mathrm{min}}(P, Q) = \inf((\mathrm{dist}_{\mathrm{min}}(P))^{\circ}Q)$.

The functor $\mathrm{dist}^{\mathrm{max}}_{\mathrm{min}}(P, Q)$ yielding a real number is defined as follows:

(Def.10) $\mathrm{dist}^{\mathrm{max}}_{\mathrm{min}}(P, Q) = \sup((\mathrm{dist}_{\mathrm{min}}(P))^{\circ}Q)$.

The functor $\mathrm{dist}^{\mathrm{min}}_{\mathrm{max}}(P, Q)$ yielding a real number is defined as follows:

(Def.11) $\mathrm{dist}^{\mathrm{min}}_{\mathrm{max}}(P, Q) = \inf((\mathrm{dist}_{\mathrm{max}}(P))^{\circ}Q)$.

The functor $\mathrm{dist}^{\mathrm{max}}_{\mathrm{max}}(P, Q)$ yielding a real number is defined as follows:

(Def.12) $\mathrm{dist}^{\mathrm{max}}_{\mathrm{max}}(P, Q) = \sup((\mathrm{dist}_{\mathrm{max}}(P))^{\circ}Q)$.

One can prove the following propositions:

(36) Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exist points x_1, x_2 of M such that $x_1 \in P$ and $x_2 \in Q$ and $\rho(x_1, x_2) = \mathrm{dist}^{\mathrm{min}}_{\mathrm{min}}(P, Q)$.

(37) Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exist points x_1, x_2 of M such that $x_1 \in P$ and $x_2 \in Q$ and $\rho(x_1, x_2) = \mathrm{dist}^{\mathrm{min}}_{\mathrm{max}}(P, Q)$.

(38) Let M be a metric space and let P, Q be subsets of the carrier of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exist points x_1, x_2 of M such that $x_1 \in P$ and $x_2 \in Q$ and $\rho(x_1, x_2) = {\rm dist}^{\rm max}_{\rm min}(P, Q)$.

(39) Let M be a metric space and let P, Q be subsets of the carrier of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exist points x_1, x_2 of M such that $x_1 \in P$ and $x_2 \in Q$ and $\rho(x_1, x_2) = {\rm dist}^{\rm max}_{\rm max}(P, Q)$.

(40) Let M be a metric space and let P, Q be subsets of the carrier of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Let x_1, x_2 be points of M. If $x_1 \in P$ and $x_2 \in Q$, then ${\rm dist}^{\rm min}_{\rm min}(P, Q) \leq \rho(x_1, x_2)$ and $\rho(x_1, x_2) \leq {\rm dist}^{\rm max}_{\rm max}(P, Q)$.

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Received July 10, 1995