FORMALIZED MATHEMATICS
Volume 5, Number 4, 1996
Warsaw University - Białystok

Examples of Category Structures

Andrzej Trybulec
Warsaw University
Białystok

Summary. We continue the formalization of the category theory.

MML Identifier: ALTCAT_2.

The notation and terminology used here are introduced in the following papers: [17], [19], [9], [20], [18], [5], [6], [2], [13], [1], [8], [4], [3], [7], [16], [12], [14], [15], [10], and [11].

1. Preliminaries

One can prove the following proposition

(1) For all sets X_1, X_2 and for arbitrary a_1, a_2 holds $[\!:X_1 \longmapsto a_1, X_2 \longmapsto a_2:\!] = [\!:X_1, X_2:\!] \longmapsto \langle a_1, a_2 \rangle$.

Let I be a set. Observe that $\emptyset_I$ is function yielding.

The following two propositions are true:

(2) For all functions f, g holds $\curvearrowleft(g \cdot f) = g \cdot \curvearrowleft f$.

(3) For all functions f, g, h holds $\curvearrowleft(f \cdot [\!:g, h:\!]) = \curvearrowleft f \cdot [\!:h, g:\!]$.

Let f be a function yielding function. Observe that $\curvearrowleft f$ is function yielding.

One can prove the following proposition

(4) Let I be a set and let A, B, C be many sorted sets indexed by I. Suppose A is transformable to B. Let F be a many sorted function from A into B and let G be a many sorted function from B into C. Then $G \circ F$ is a many sorted function from A into C.

Let I be a set and let A be a many sorted set indexed by $[\!:I, I:\!]$. Then $\curvearrowleft A$ is a many sorted set indexed by $[\!:I, I:\!]$.

We now state the proposition

ISSN 1426–2630

(5) Let I_1 be a set, and let I_2 be a non empty set, and let f be a function from I_1 into I_2, and let B, C be many sorted sets indexed by I_2, and let G be a many sorted function from B into C. Then $G \cdot f$ is a many sorted function from $B \cdot f$ into $C \cdot f$.

Let I be a set, let A, B be many sorted sets indexed by $[\!: I, I :\!]$, and let F be a many sorted function from A into B. Then $\curvearrowleft\!F$ is a many sorted function from $\curvearrowleft\!A$ into $\curvearrowleft\!B$.

We now state the proposition

(6) Let I_1, I_2 be non empty sets, and let M be a many sorted set indexed by $[\!: I_1, I_2 :\!]$ and let o_1 be an element of I_1, and let o_2 be an element of I_2. Then $(\curvearrowleft\!M)(o_2, o_1) = M(o_1, o_2)$.

Let I_1 be a set and let f, g be many sorted functions of I_1 Then $g \circ f$ is a many sorted function of I_1.

2. An auxiliary notion

Let I, J be sets, let A be a many sorted set indexed by I, and let B be a many sorted set indexed by J. The predicate $A \mathrel{\dot{\subseteq}} B$ is defined as follows:

(Def. 1) $I \subseteq J$ and for arbitrary i such that $i \in I$ holds $A(i) \subseteq B(i)$.

One can prove the following four propositions:

(7) For every set I and for every many sorted set A indexed by I holds $A \mathrel{\dot{\subseteq}} A$.

(8) Let I, J be sets, and let A be a many sorted set indexed by I, and let B be a many sorted set indexed by J. If $A \mathrel{\dot{\subseteq}} B$ and $B \mathrel{\dot{\subseteq}} A$, then $A = B$.

(9) Let I, J, K be sets, and let A be a many sorted set indexed by I, and let B be a many sorted set indexed by J, and let C be a many sorted set indexed by K. If $A \mathrel{\dot{\subseteq}} B$ and $B \mathrel{\dot{\subseteq}} C$, then $A \mathrel{\dot{\subseteq}} C$.

(10) Let I be a set, and let A be a many sorted set indexed by I, and let B be a many sorted set indexed by I. Then $A \mathrel{\dot{\subseteq}} B$ if and only if $A \subseteq B$.

3. A bit of lambda calculus

In this article we present several logical schemes. The scheme *OnSingletons* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding arbitrary, and a unary predicate $\mathcal{P}$, and states that:

$\{\langle o, \mathcal{F}(o)\rangle : o$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[o]\}$ is a function

for all values of the parameters.

The scheme *DomOnSingletons* deals with a non empty set $\mathcal{A}$, a function $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding arbitrary, and a unary predicate $\mathcal{P}$, and states that:

$\operatorname{dom}\mathcal{B} = \{o : o$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[o]\}$

provided the following condition is satisfied:

- $\mathcal{B} = \{\langle o, \mathcal{F}(o)\rangle : o$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[o]\}$.

The scheme *ValOnSingletons* deals with a non empty set $\mathcal{A}$, a function $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding arbitrary, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{B}(\mathcal{C}) = \mathcal{F}(\mathcal{C})$

provided the following requirements are met:

- $\mathcal{B} = \{\langle o, \mathcal{F}(o)\rangle : o$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[o]\}$,
- $\mathcal{P}[\mathcal{C}]$.

4. More on old categories

The following propositions are true:

(11) For every category C and for all objects i, j, k of C holds $[\![\hom(j,k), \hom(i,j)]\!] \subseteq \operatorname{dom}$ (the composition of C).

(12) For every category C and for all objects i, j, k of C holds (the composition of C)$^\circ [\![\hom(j,k), \hom(i,j)]\!] \subseteq \hom(i,k)$.

Let C be a category structure. The functor $\mathrm{HomSets}_C$ yields a many sorted set indexed by $[\![$ the objects of C, the objects of $C]\!]$ and is defined as follows:

(Def. 2) For all objects i, j of C holds $\mathrm{HomSets}_C(i, j) = \hom(i,j)$.

The following proposition is true

(13) For every category C and for every object i of C holds $\mathrm{id}_i \in \mathrm{HomSets}_C(i, i)$.

Let C be a category. The functor $\mathrm{Composition}_C$ yielding a binary composition of $\mathrm{HomSets}_C$ is defined by:

(Def. 3) For all objects i, j, k of C holds $\mathrm{Composition}_C(i, j, k) =$ (the composition of C) $\restriction [\![\mathrm{HomSets}_C(j, k), \mathrm{HomSets}_C(i, j)]\!]$.

Next we state three propositions:

(14) Let C be a category and let i, j, k be objects of C Suppose $\hom(i,j) \neq \emptyset$ and $\hom(j,k) \neq \emptyset$. Let f be a morphism from i to j and let g be a morphism from j to k. Then $\mathrm{Composition}_C(i, j, k)(g, f) = g \cdot f$.

(15) For every category C holds $\mathrm{Composition}_C$ is associative.

(16) For every category C holds $\mathrm{Composition}_C$ has left units and right units.

5. Transforming an old category into a new one

Let C be a category. The functor $\mathrm{Alter}(C)$ yielding a strict non empty category structure is defined as follows:

(Def. 4) $\mathrm{Alter}(C) = \langle$the objects of C, $\mathrm{HomSets}_C, \mathrm{Composition}_C\rangle$.

We now state three propositions:

(17) For every category C holds $\mathrm{Alter}(C)$ is associative.

(18) For every category C holds $\mathrm{Alter}(C)$ has units.

(19) For every category C holds $\mathrm{Alter}(C)$ is transitive.

Let C be a category. Then $\mathrm{Alter}(C)$ is a strict category.

6. More on new categories

Let us note that there exists a graph which is non empty and strict.

Let C be a graph. We say that C is reflexive if and only if:

(Def. 5) For arbitrary x such that $x \in$ the carrier of C holds (the arrows of C)(x, x) $\neq \emptyset$.

Let C be a non empty graph. Let us observe that C is reflexive if and only if:

(Def. 6) For every object o of C holds $\langle o, o \rangle \neq \emptyset$.

Let C be a non empty category structure. Observe that the carrier of C is non empty.

Let C be a non empty transitive category structure. Let us observe that C is associative if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let o_1, o_2, o_3, o_4 be objects of C and let f be a morphism from o_1 to o_2, and let g be a morphism from o_2 to o_3, and let h be a morphism from o_3 to o_4. If $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_3, o_4 \rangle \neq \emptyset$, then $(h \cdot g) \cdot f = h \cdot (g \cdot f)$.

Let C be a non empty category structure. Let us observe that C has units if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let o be an object of C. Then

(i) $\langle o, o \rangle \neq \emptyset$, and

(ii) there exists a morphism i from o to o such that for every object o' of C and for every morphism m' from o' to o and for every morphism m'' from o to o' holds if $\langle o', o \rangle \neq \emptyset$, then $i \cdot m' = m'$ and if $\langle o, o' \rangle \neq \emptyset$, then $m'' \cdot i = m''$.

Let us observe that every non empty category structure which has units is reflexive.

One can check that there exists a graph which is non empty and reflexive.

One can verify that there exists a category structure which is non empty and reflexive.

7. The empty category

The strict category structure $\emptyset_{CAT}$ is defined by:

(Def. 9) The carrier of $\emptyset_{CAT}$ is empty.

Let us note that $\emptyset_{CAT}$ is empty.

Let us mention that there exists a category structure which is empty and strict.

Next we state the proposition

(20) For every empty strict category structure E holds $E = \emptyset_{CAT}$.

8. Subcategories

Let C be a category structure. A category structure is said to be a substructure of C if it satisfies the conditions (Def. 10).

(Def. 10) (i) The carrier of it $\subseteq$ the carrier of C,

(ii) the arrows of it $\dot{\subseteq}$ the arrows of C, and

(iii) the composition of it $\dot{\subseteq}$ the composition of C.

In the sequel C, C_1, C_2, C_3 denote category structures.

The following propositions are true:

(21) C is a substructure of C.

(22) If C_1 is a substructure of C_2 and C_2 is a substructure of C_3, then C_1 is a substructure of C_3.

(23) Let C_1, C_2 be category structures. Suppose C_1 is a substructure of C_2 and C_2 is a substructure of C_1. Then the category structure of C_1 = the category structure of C_2.

Let C be a category structure. One can check that there exists a substructure of C which is strict.

Let C be a non empty category structure and let o be an object of C. The functor $\Box{\restriction}o$ yielding a strict substructure of C is defined by the conditions (Def. 11).

(Def. 11) (i) The carrier of $\Box{\restriction}o = \{o\}$,

(ii) the arrows of $\Box{\restriction}o = [\langle o, o\rangle \mapsto \langle o, o\rangle]$, and

(iii) the composition of $\Box{\restriction}o = \langle o, o, o\rangle \dot{\longmapsto}$(the composition of C)(o, o, o).

In the sequel C denotes a non empty category structure and o denotes an object of C.

One can prove the following proposition

(24) For every object o' of $\Box{\restriction}o$ holds $o' = o$.

Let C be a non empty category structure and let o be an object of C. Observe that $\Box{\restriction}o$ is transitive and non empty.

Let C be a non empty category structure. One can verify that there exists a substructure of C which is transitive non empty and strict.

We now state the proposition

(25) Let C be a transitive non empty category structure and let D_1, D_2 be transitive non empty substructures of C. Suppose the carrier of $D_1 \subseteq$ the carrier of D_2 and the arrows of $D_1 \dot{\subseteq}$ the arrows of D_2. Then D_1 is a substructure of D_2.

Let C be a category structure and let D be a substructure of C. We say that D is full if and only if:

(Def. 12) The arrows of $D =$ (the arrows of C) $\restriction$ [: the carrier of D, the carrier of D :].

Let C be a non empty category structure with units and let D be a substructure of C. We say that D is id-inheriting if and only if:

(Def. 13) For every object o of D and for every object o' of C such that $o = o'$ holds $\mathrm{id}_{o'} \in \langle o, o \rangle$.

Let C be a category structure. One can verify that there exists a substructure of C which is full and strict.

Let C be a non empty category structure. Observe that there exists a substructure of C which is full non empty and strict.

Let C be a category and let o be an object of C. Note that $\Box\restriction o$ is full and id-inheriting.

Let C be a category. One can verify that there exists a substructure of C which is full id-inheriting non empty and strict.

In the sequel C is a non empty transitive category structure.

The following propositions are true:

(26) Let D be a substructure of C. Suppose the carrier of $D =$ the carrier of C and the arrows of $D =$ the arrows of C. Then the category structure of $D =$ the category structure of C.

(27) Let D_1, D_2 be non empty transitive substructures of C. Suppose the carrier of $D_1 =$ the carrier of D_2 and the arrows of $D_1 =$ the arrows of D_2. Then the category structure of $D_1 =$ the category structure of D_2.

(28) Let D be a full substructure of C. Suppose the carrier of $D =$ the carrier of C. Then the category structure of $D =$ the category structure of C.

(29) Let C be a non empty category structure, and let D be a full non empty substructure of C, and let o_1, o_2 be objects of C and let p_1, p_2 be objects of D If $o_1 = p_1$ and $o_2 = p_2$, then $\langle o_1, o_2 \rangle = \langle p_1, p_2 \rangle$.

(30) For every non empty category structure C and for every non empty substructure D of C holds every object of D is an object of C.

Let C be a transitive non empty category structure. Note that every substructure of C which is full and non empty is also transitive.

The following propositions are true:

(31) Let D_1, D_2 be full non empty substructures of C. Suppose the carrier of D_1 = the carrier of D_2. Then the category structure of D_1 = the category structure of D_2.

(32) Let C be a non empty category structure, and let D be a non empty substructure of C, and let o_1, o_2 be objects of C and let p_1, p_2 be objects of D If $o_1 = p_1$ and $o_2 = p_2$, then $\langle p_1, p_2 \rangle \subseteq \langle o_1, o_2 \rangle$.

(33) Let C be a non empty transitive category structure, and let D be a non empty transitive substructure of C, and let p_1, p_2, p_3 be objects of D Suppose $\langle p_1, p_2 \rangle \neq \emptyset$ and $\langle p_2, p_3 \rangle \neq \emptyset$. Let o_1, o_2, o_3 be objects of C Suppose $o_1 = p_1$ and $o_2 = p_2$ and $o_3 = p_3$. Let f be a morphism from o_1 to o_2, and let g be a morphism from o_2 to o_3, and let f_1 be a morphism from p_1 to p_2, and let g_1 be a morphism from p_2 to p_3. If $f = f_1$ and $g = g_1$, then $g \cdot f = g_1 \cdot f_1$.

Let C be an associative transitive non empty category structure. Note that every non empty substructure of C which is transitive is also associative.

One can prove the following proposition

(34) Let C be a non empty category structure, and let D be a non empty substructure of C, and let o_1, o_2 be objects of C and let p_1, p_2 be objects of D If $o_1 = p_1$ and $o_2 = p_2$ and $\langle p_1, p_2 \rangle \neq \emptyset$, then every morphism from p_1 to p_2 is a morphism from o_1 to o_2.

Let C be a transitive non empty category structure with units. Note that every non empty substructure of C which is id-inheriting and transitive has units.

Let C be a category. Note that there exists a non empty substructure of C which is id-inheriting and transitive.

Let C be a category. A subcategory of C is an id-inheriting transitive substructure of C.

We now state the proposition

(35) Let C be a category, and let D be a non empty subcategory of C, and let o be an object of D, and let o' be an object of C. If $o = o'$, then $\mathrm{id}_o = \mathrm{id}_{o'}$.

References

[1] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[3] Czesław Byliński. Cartesian categories. *Formalized Mathematics*, 3(**2**):161–169, 1992.

[4] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[7] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(**2**):409–420, 1990.

[8] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[9] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[10] Artur Korniłowicz. On the group of automorphisms of universal algebra & many sorted algebra. *Formalized Mathematics*, 5(**2**):221–226, 1996.
[11] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Formalized Mathematics*, 5(**1**):61–65, 1996.
[12] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.
[13] Michał Muzalewski and Wojciech Skaba. Three-argument operations and four-argument operations. *Formalized Mathematics*, 2(**2**):221–224, 1991.
[14] Andrzej Trybulec. Categories without uniqueness of **cod** and **dom**. *Formalized Mathematics*, 5(**2**):259–267, 1996.
[15] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.
[16] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[18] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[19] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[20] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received January 22, 1996