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Miscellaneous Facts about Functions

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The papers [16], [26], [3], [24], [29], [14], [28], [19], [23], [25], [22], [1], [17], [18], [30], [10], [6], [5], [15], [8], [13], [7], [11], [21], [9], [12], [2], [27], [20], and [4] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity we adopt the following rules: x is arbitrary, m, n are natural numbers, f, g are functions, and A, B are sets.

We now state several propositions:

(1) For every function f and for every set X such that $\operatorname{rng} f \subseteq X$ holds $\mathrm{id}_X \cdot f = f$.

(2) Let X be a set, and let Y be a non empty set, and let f be a function from X into Y. Suppose f is one-to-one. Let B be a subset of X and let C be a subset of Y. If $C \subseteq f^\circ B$, then $f^{-1} C \subseteq B$.

(3) Let X, Y be non empty sets and let f be a function from X into Y. Suppose f is one-to-one. Let x be an element of X and let A be a subset of X. If $f(x) \in f^\circ A$, then $x \in A$.

(4) Let X, Y be non empty sets and let f be a function from X into Y. Suppose f is one-to-one. Let x be an element of X, and let A be a subset of X, and let B be a subset of Y. If $f(x) \in f^\circ A \setminus B$, then $x \in A \setminus f^{-1} B$.

(5) Let X, Y be non empty sets and let f be a function from X into Y. Suppose f is one-to-one. Let y be an element of Y, and let A be a subset of X, and let B be a subset of Y. If $y \in f^\circ A \setminus B$, then $f^{-1}(y) \in A \setminus f^{-1} B$.

(6) For every function f and for arbitrary a such that $a \in \operatorname{dom} f$ holds $f \upharpoonright \{a\} = a \longmapsto f(a)$.

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Let x, y be arbitrary. Observe that $x \stackrel{.}{\longmapsto} y$ is non empty.

Let x, y, a, b be arbitrary. One can check that $[x \longmapsto a, y \longmapsto b]$ is non empty.

One can prove the following propositions:

(7) For every set I and for every many sorted set M indexed by I and for arbitrary i such that $i \in I$ holds $i \stackrel{.}{\longmapsto} M(i) = M \restriction \{i\}$.

(8) Let I, J be sets, and let M be a many sorted set indexed by $[: I, J :]$, and let i, j be arbitrary. If $i \in I$ and $j \in J$, then $[\langle i, j \rangle \mapsto M(i, j)] = M \restriction [: \{i\}, \{j\} :]$.

(9) If $x \in \operatorname{dom} f$ and $x \notin \operatorname{dom} g$, then $(f {+}\cdot g)(x) = f(x)$.

(10) For all functions f, g, h such that $\operatorname{rng} g \subseteq \operatorname{dom} f$ and $\operatorname{rng} h \subseteq \operatorname{dom} f$ holds $f \cdot (g {+}\cdot h) = f \cdot g {+}\cdot f \cdot h$.

(11) For all functions f, g, h holds $(g {+}\cdot h) \cdot f = g \cdot f {+}\cdot h \cdot f$.

(12) For all functions f, g, h such that $\operatorname{rng} f$ misses $\operatorname{dom} g$ holds $(h {+}\cdot g) \cdot f = h \cdot f$.

(13) For all sets A, B and for arbitrary y such that A meets $\operatorname{rng}(\mathrm{id}_B {+}\cdot (A \longmapsto y))$ holds $y \in A$.

(14) For arbitrary x, y and for every set A such that $x \neq y$ holds $x \notin \operatorname{rng}(\mathrm{id}_A {+}\cdot (x \stackrel{.}{\longmapsto} y))$.

(15) For every set X and for arbitrary a and for every function f such that $\operatorname{dom} f = X \cup \{a\}$ holds $f = f \restriction X {+}\cdot (a \stackrel{.}{\longmapsto} f(a))$.

(16) For every function f and for all sets X, y, z holds $f {+}\cdot (X \longmapsto y) {+}\cdot (X \longmapsto z) = f {+}\cdot (X \longmapsto z)$.

(17) If $0 < m$ and $m \leq n$, then $\mathbb{Z}_m \subseteq \mathbb{Z}_n$.

(18) $\mathbb{Z} \neq \mathbb{Z}^*$.

(19) $\emptyset^* = \{\emptyset\}$.

(20) $\langle x \rangle \in A^*$ iff $x \in A$.

(21) $A \subseteq B$ iff $A^* \subseteq B^*$.

(22) For every subset A of $\mathbb{N}$ such that for all n, m such that $n \in A$ and $m < n$ holds $m \in A$ holds A is a cardinal number.

(23) Let A be a finite set and let X be a non empty family of subsets of A. Then there exists an element C of X such that for every element B of X such that $B \subseteq C$ holds $B = C$.

(24) Let p, q be finite sequences. Suppose $\operatorname{len} p = \operatorname{len} q + 1$. Let i be a natural number. Then $i \in \operatorname{dom} q$ if and only if the following conditions are satisfied:

  (i) $i \in \operatorname{dom} p$, and

  (ii) $i + 1 \in \operatorname{dom} p$.

Let us note that there exists a finite sequence which is function yielding non empty and non-empty.

Note that ε is function yielding. Let f be a function. Observe that $\langle f \rangle$ is function yielding. Let g be a function. One can check that $\langle f, g \rangle$ is function yielding. Let h be a function. Observe that $\langle f, g, h\rangle$ is function yielding.

Let n be a natural number and let f be a function. One can verify that $n \mapsto f$ is function yielding.

Let p be a finite sequence and let q be a non empty finite sequence. One can verify that $p \frown q$ is non empty and $q \frown p$ is non empty.

Let p, q be function yielding finite sequences. Note that $p \frown q$ is function yielding.

Next we state the proposition

(25) Let p, q be finite sequences. Suppose $p \frown q$ is function yielding. Then p is function yielding and q is function yielding.

2. Some useful schemes

In this article we present several logical schemes. The scheme *KappaD* concerns non empty sets $\mathcal{A}$, $\mathcal{B}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

There exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

provided the parameters meet the following condition:

- For every element x of $\mathcal{A}$ holds $\mathcal{F}(x) \in \mathcal{B}$.

The scheme *Kappa2D* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a binary functor $\mathcal{F}$ yielding arbitrary, and states that:

There exists a function f from $[:\mathcal{A}, \mathcal{B}:]$ into $\mathcal{C}$ such that for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $f(\langle x, y\rangle) = \mathcal{F}(x, y)$

provided the parameters meet the following requirement:

- For every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\mathcal{F}(x, y) \in \mathcal{C}$.

The scheme *FinMono* concerns a set $\mathcal{A}$, a non empty set $\mathcal{B}$, and two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding arbitrary, and states that:

$\{\mathcal{F}(d) : d$ ranges over elements of $\mathcal{B}$, $\mathcal{G}(d) \in \mathcal{A}\}$ is finite

provided the following conditions are satisfied:

- $\mathcal{A}$ is finite,
- For all elements d_1, d_2 of $\mathcal{B}$ such that $\mathcal{G}(d_1) = \mathcal{G}(d_2)$ holds $d_1 = d_2$.

The scheme *CardMono* concerns a set $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

$\mathcal{A} \approx \{d : d$ ranges over elements of $\mathcal{B}$, $\mathcal{F}(d) \in \mathcal{A}\}$

provided the following requirements are met:

- For arbitrary x such that $x \in \mathcal{A}$ there exists an element d of $\mathcal{B}$ such that $x = \mathcal{F}(d)$,
- For all elements d_1, d_2 of $\mathcal{B}$ such that $\mathcal{F}(d_1) = \mathcal{F}(d_2)$ holds $d_1 = d_2$.

The scheme *CardMono'* concerns a set $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

$\mathcal{A} \approx \{\mathcal{F}(d) : d$ ranges over elements of $\mathcal{B}$, $d \in \mathcal{A}\}$

provided the following conditions are satisfied:

- $\mathcal{A} \subseteq \mathcal{B}$,
- For all elements d_1, d_2 of $\mathcal{B}$ such that $\mathcal{F}(d_1) = \mathcal{F}(d_2)$ holds $d_1 = d_2$.

The scheme *FuncSeqInd* concerns a unary predicate $\mathcal{P}$, and states that:

For every function yielding finite sequence p holds $\mathcal{P}[p]$

provided the following conditions are satisfied:

- $\mathcal{P}[\varepsilon]$,
- For every function yielding finite sequence p such that $\mathcal{P}[p]$ and for every function f holds $\mathcal{P}[p \frown \langle f \rangle]$.

3. Some auxiliary concepts

Let x be arbitrary and let y be a set. Let us assume that $x \in y$. The functor $x(\in y)$ yielding an element of y is defined as follows:

(Def. 1) $x(\in y) = x$.

One can prove the following proposition

(26) If $x \in A \cap B$, then $x(\in A) = x(\in B)$.

Let f, g be functions and let A be a set. We say that f and g equal outside A if and only if:

(Def. 2) $f \restriction (\operatorname{dom} f \setminus A) = g \restriction (\operatorname{dom} g \setminus A)$.

Next we state several propositions:

(27) For every function f and for every set A holds f and f equal outside A.

(28) For all functions f, g and for every set A such that f and g equal outside A holds g and f equal outside A

(29) Let f, g, h be functions and let A be a set. Suppose f and g equal outside A and g and h equal outside A. Then f and h equal outside A.

(30) For all functions f, g and for every set A such that f and g equal outside A holds $\operatorname{dom} f \setminus A = \operatorname{dom} g \setminus A$.

(31) For all functions f, g and for every set A such that $\operatorname{dom} g \subseteq A$ holds f and $f{+\cdot}g$ equal outside A

Let f be a function and let i, x be arbitrary. The functor $f +\cdot (i,x)$ yields a function and is defined by:

(Def. 3) (i) $f +\cdot (i,x) = f{+\cdot}(i \mapsto x)$ if $i \in \operatorname{dom} f$,

(ii) $f +\cdot (i,x) = f$, otherwise.

Next we state several propositions:

(32) For every function f and for arbitrary d, i holds $\operatorname{dom}(f +\cdot (i,d)) = \operatorname{dom} f$.

(33) For every function f and for arbitrary d, i such that $i \in \operatorname{dom} f$ holds $(f +\cdot (i,d))(i) = d$.

(34) For every function f and for arbitrary d, i, j such that $i \neq j$ and $j \in \operatorname{dom} f$ holds $(f +\cdot (i,d))(j) = f(j)$.

(35) For every function f and for arbitrary d, e, i, j such that $i \neq j$ holds $f +\cdot (i,d) +\cdot (j,e) = f +\cdot (j,e) +\cdot (i,d)$.

(36) For every function f and for arbitrary d, e, i holds $f +\cdot (i,d) +\cdot (i,e) = f +\cdot (i,e)$.

(37) For every function f and for arbitrary i holds $f +\cdot (i, f(i)) = f$.

Let f be a finite sequence, let i be a natural number, and let x be arbitrary. One can check that $f +\cdot (i,x)$ is finite sequence-like.

Let D be a set, let f be a finite sequence of elements of D, let i be a natural number, and let d be an element of D. Then $f +\cdot (i,d)$ is a finite sequence of elements of D.

The following three propositions are true:

(38) Let D be a non empty set, and let f be a finite sequence of elements of D, and let d be an element of D, and let i be a natural number. If $i \in \operatorname{dom} f$, then $\pi_i(f +\cdot (i,d)) = d$.

(39) Let D be a non empty set, and let f be a finite sequence of elements of D, and let d be an element of D, and let i, j be natural numbers. If $i \neq j$ and $j \in \operatorname{dom} f$, then $\pi_j(f +\cdot (i,d)) = \pi_j f$.

(40) Let D be a non empty set, and let f be a finite sequence of elements of D, and let d, e be elements of D, and let i be a natural number. Then $f +\cdot (i, \pi_i f) = f$.

4. On the composition of a finite sequence of functions

Let X be a set and let p be a function yielding finite sequence. The functor $\operatorname{compose}_X p$ yielding a function is defined by the condition (Def. 4).

(Def. 4) There exists a many sorted function f of $\mathbb{N}$ such that

(i) $\operatorname{compose}_X p = f(\operatorname{len} p)$,

(ii) $f(0) = \operatorname{id}_X$, and

(iii) for every natural number i such that $i+1 \in \operatorname{dom} p$ and for all functions g, h such that $g = f(i)$ and $h = p(i+1)$ holds $f(i+1) = h \cdot g$.

Let p be a function yielding finite sequence and let x be a set. The functor $\operatorname{apply}(p,x)$ yields a finite sequence and is defined by the conditions (Def. 5).

(Def. 5) (i) $\operatorname{len} \operatorname{apply}(p,x) = \operatorname{len} p + 1$,

(ii) $(\operatorname{apply}(p,x))(1) = x$, and

(iii) for every natural number i and for every function f such that $i \in \operatorname{dom} p$ and $f = p(i)$ holds $(\operatorname{apply}(p,x))(i+1) = f((\operatorname{apply}(p,x))(i))$.

We adopt the following convention: X, Y, x denote sets, p, q denote function yielding finite sequences, and f, g, h denote functions.

The following propositions are true:

(41) $\mathrm{compose}_X\,\varepsilon = \mathrm{id}_X$.

(42) $\mathrm{apply}(\varepsilon, x) = \langle x \rangle$.

(43) $\mathrm{compose}_X(p \frown \langle f \rangle) = f \cdot \mathrm{compose}_X\,p$.

(44) $\mathrm{apply}(p \frown \langle f \rangle, x) = (\mathrm{apply}(p, x)) \frown \langle f((\mathrm{apply}(p, x))(\mathrm{len}\,p + 1)) \rangle$.

(45) $\mathrm{compose}_X(\langle f \rangle \frown p) = \mathrm{compose}_{f^\circ X}\,p \cdot (f \restriction X)$.

(46) $\mathrm{apply}(\langle f \rangle \frown p, x) = \langle x \rangle \frown \mathrm{apply}(p, f(x))$.

(47) $\mathrm{compose}_X\langle f \rangle = f \cdot \mathrm{id}_X$.

(48) If $\mathrm{dom}\,f \subseteq X$, then $\mathrm{compose}_X\langle f \rangle = f$.

(49) $\mathrm{apply}(\langle f \rangle, x) = \langle x, f(x) \rangle$.

(50) If $\mathrm{rng}\,\mathrm{compose}_X\,p \subseteq Y$, then $\mathrm{compose}_X(p \frown q) = \mathrm{compose}_Y\,q \cdot \mathrm{compose}_X\,p$.

(51) $(\mathrm{apply}(p \frown q, x))(\mathrm{len}(p \frown q) + 1) = (\mathrm{apply}(q, (\mathrm{apply}(p, x))(\mathrm{len}\,p + 1)))(\mathrm{len}\,q + 1)$.

(52) $\mathrm{apply}(p \frown q, x) = (\mathrm{apply}(p, x)) \ ^{\$}\!\!\frown \mathrm{apply}(q, (\mathrm{apply}(p, x))(\mathrm{len}\,p + 1))$.

(53) $\mathrm{compose}_X\langle f, g \rangle = g \cdot f \cdot \mathrm{id}_X$.

(54) If $\mathrm{dom}\,f \subseteq X$ or $\mathrm{dom}(g \cdot f) \subseteq X$, then $\mathrm{compose}_X\langle f, g \rangle = g \cdot f$.

(55) $\mathrm{apply}(\langle f, g \rangle, x) = \langle x, f(x), g(f(x)) \rangle$.

(56) $\mathrm{compose}_X\langle f, g, h \rangle = h \cdot g \cdot f \cdot \mathrm{id}_X$.

(57) If $\mathrm{dom}\,f \subseteq X$ or $\mathrm{dom}(g \cdot f) \subseteq X$ or $\mathrm{dom}(h \cdot g \cdot f) \subseteq X$, then $\mathrm{compose}_X\langle f, g, h \rangle = h \cdot g \cdot f$.

(58) $\mathrm{apply}(\langle f, g, h \rangle, x) = \langle x \rangle \frown \langle f(x), g(f(x)), h(g(f(x))) \rangle$.

Let F be a finite sequence. The functor $\mathrm{firstdom}(F)$ is defined as follows:

(Def. 6) (i) $\mathrm{firstdom}(F)$ is empty if F is empty,

(ii) $\mathrm{firstdom}(F) = \pi_1(F(1))$, otherwise.

The functor $\mathrm{lastrng}(F)$ is defined by:

(Def. 7) (i) $\mathrm{lastrng}(F)$ is empty if F is empty,

(ii) $\mathrm{lastrng}(F) = \pi_2(F(\mathrm{len}\,F))$, otherwise.

Next we state three propositions:

(59) $\mathrm{firstdom}(\varepsilon) = \emptyset$ and $\mathrm{lastrng}(\varepsilon) = \emptyset$.

(60) For every finite sequence p holds $\mathrm{firstdom}(\langle f \rangle \frown p) = \mathrm{dom}\,f$ and $\mathrm{lastrng}(p \frown \langle f \rangle) = \mathrm{rng}\,f$.

(61) For every function yielding finite sequence p such that $p \neq \varepsilon$ holds $\mathrm{rng}\,\mathrm{compose}_X\,p \subseteq \mathrm{lastrng}(p)$.

Let I_1 be a finite sequence. We say that I_1 is composable if and only if:

(Def. 8) There exists a finite sequence p such that $\mathrm{len}\,p = \mathrm{len}\,I_1 + 1$ and for every natural number i such that $i \in \mathrm{dom}\,I_1$ holds $I_1(i) \in p(i+1)^{p(i)}$.

We now state the proposition

(62) For all finite sequences p, q such that $p \frown q$ is composable holds p is composable and q is composable.

One can verify that every finite sequence which is composable is also function yielding.

Let us observe that every finite sequence which is empty is also composable.

Let f be a function. One can check that $\langle f \rangle$ is composable.

Let us observe that there exists a finite sequence which is composable non empty and non-empty.

A composable sequence is a composable finite sequence.

Next we state several propositions:

(63) For every composable sequence p such that $p \neq \varepsilon$ holds $\mathrm{dom}\,\mathrm{compose}_X\, p = \mathrm{firstdom}(p) \cap X$.

(64) For every composable sequence p holds $\mathrm{dom}\,\mathrm{compose}_{\mathrm{firstdom}(p)}\, p = \mathrm{firstdom}(p)$.

(65) For every composable sequence p and for every function f such that $\mathrm{rng}\, f \subseteq \mathrm{firstdom}(p)$ holds $\langle f \rangle \frown p$ is a composable sequence.

(66) For every composable sequence p and for every function f such that $\mathrm{lastrng}(p) \subseteq \mathrm{dom}\, f$ holds $p \frown \langle f \rangle$ is a composable sequence.

(67) For every composable sequence p such that $x \in \mathrm{firstdom}(p)$ and $x \in X$ holds $(\mathrm{apply}(p, x))(\mathrm{len}\, p + 1) = (\mathrm{compose}_X\, p)(x)$.

Let X, Y be sets. Let us assume that if Y is empty, then X is empty. A composable sequence is called a composable sequence from X into Y if:

(Def. 9) $\mathrm{firstdom}(\mathrm{it}) = X$ and $\mathrm{lastrng}(\mathrm{it}) \subseteq Y$.

Let Y be a non empty set, let X be a set, and let F be a composable sequence from X into Y. Then $\mathrm{compose}_X\, F$ is a function from X into Y.

Let q be a non-empty non empty finite sequence. A finite sequence is said to be a composable sequence along q if:

(Def. 10) $\mathrm{len}\,\mathrm{it} + 1 = \mathrm{len}\, q$ and for every natural number i such that $i \in \mathrm{dom}\,\mathrm{it}$ holds $\mathrm{it}(i) \in q(i+1)^{q(i)}$.

Let q be a non-empty non empty finite sequence. Observe that every composable sequence along q is composable and non-empty.

One can prove the following three propositions:

(68) Let q be a non-empty non empty finite sequence and let p be a composable sequence along q. If $p \neq \varepsilon$, then $\mathrm{firstdom}(p) = q(1)$ and $\mathrm{lastrng}(p) \subseteq q(\mathrm{len}\, q)$.

(69) Let q be a non-empty non empty finite sequence and let p be a composable sequence along q. Then $\mathrm{dom}\,\mathrm{compose}_{q(1)}\, p = q(1)$ and $\mathrm{rng}\,\mathrm{compose}_{q(1)}\, p \subseteq q(\mathrm{len}\, q)$.

(70) For every function f and for every natural number n holds $f^n = \mathrm{compose}_{\mathrm{dom}\, f \cup \mathrm{rng}\, f}(n \mapsto f)$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.
[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[4] Grzegorz Bancerek. Reduction relations. *Formalized Mathematics*, 5(**4**):469–478, 1996.
[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[6] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[7] Czesław Byliński. Cartesian categories. *Formalized Mathematics*, 3(**2**):161–169, 1992.
[8] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.
[9] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[10] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[11] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[12] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[13] Czesław Byliński. Products and coproducts in categories. *Formalized Mathematics*, 2(**5**):701–709, 1991.
[14] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[15] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[16] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[17] Jarosław Kotowicz, Beata Madras, and Małgorzata Korolkiewicz. Basic notation of universal algebra. *Formalized Mathematics*, 3(**2**):251–253, 1992.
[18] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.
[19] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[20] Dariusz Surowik. Cyclic groups and some of their properties - part I. *Formalized Mathematics*, 2(**5**):623–627, 1991.
[21] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[22] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[23] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[24] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[25] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[26] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[27] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[28] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[29] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[30] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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