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The Correspondence Between Monotonic Many Sorted Signatures and Well-Founded Graphs. Part I [1]

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Summary. We prove a number of auxiliary facts about graphs, mainly about vertex sequences of chains and oriented chains. Then we define a graph to be *well-founded* if for each vertex in the graph the length of oriented chains ending at the vertex is bounded. A *well-founded* graph does not have directed cycles or infinite descending chains. In the second part of the article we prove some auxiliary facts about free algebras and locally-finite algebras.

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The papers [32], [34], [17], [21], [3], [1], [27], [7], [35], [14], [16], [15], [29], [19], [11], [33], [22], [24], [20], [4], [6], [8], [2], [5], [18], [12], [31], [30], [13], [23], [28], [26], [25], [9], and [10] provide the notation and terminology for this paper.

1. Some properties of graphs

The following proposition is true

(1) For every finite function f such that for every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is finite holds $\prod f$ is finite.

In the sequel G will denote a graph and m, n will denote natural numbers.

Let G be a graph. Let us note that the chain of G can be characterized by the following (equivalent) condition:

(Def. 1) It is a finite sequence of elements of the edges of G and there exists finite sequence of elements of the vertices of G which is vertex sequence of it.

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One can prove the following proposition

(2) For all finite sequences p, q such that $1 \leq n$ and $n \leq \operatorname{len} p$ holds $\langle p(1), \dots, p(n) \rangle = \langle (p \mathbin{^\frown} q)(1), \dots, (p \mathbin{^\frown} q)(n) \rangle$.

Let G be a graph and let I_1 be a chain of G. We introduce I_1 is directed as a synonym of I_1 is oriented.

Let G be a graph and let I_1 be a chain of G. We say that I_1 is cyclic if and only if:

(Def. 2) There exists a finite sequence p of elements of the vertices of G such that p is vertex sequence of I_1 and $p(1) = p(\operatorname{len} p)$.

Let I_1 be a graph. We say that I_1 is empty if and only if:

(Def. 3) The edges of I_1 is empty.

One can verify that there exists a graph which is empty.

Next we state the proposition

(3) For every graph G holds rng (the source of G) $\cup$ rng (the target of G) $\subseteq$ the vertices of G.

Let us observe that there exists a graph which is finite simple connected non empty and strict.

Let G be a non empty graph. Note that the edges of G is non empty.

We now state two propositions:

(4) Let e be arbitrary. Suppose $e \in$ the edges of G. Let s, t be elements of the vertices of G. Suppose $s =$ (the source of G)(e) and $t =$ (the target of G)(e). Then $\langle s, t \rangle$ is vertex sequence of $\langle e \rangle$.

(5) For arbitrary e such that $e \in$ the edges of G holds $\langle e \rangle$ is a directed chain of G.

In the sequel G is a non empty graph.

Let us consider G. Observe that there exists a chain of G which is directed non empty and path-like.

The following propositions are true:

(6) Let c be a chain of G and let p be a finite sequence of elements of the vertices of G. If c is cyclic and p is vertex sequence of c, then $p(1) = p(\operatorname{len} p)$.

(7) Let G be a graph and let e be arbitrary. Suppose $e \in$ the edges of G. Let f_1 be a directed chain of G. If $f_1 = \langle e \rangle$, then vertex-seq(f_1) $= \langle$(the source of G)(e), (the target of G)(e)$\rangle$.

(8) For every finite sequence f holds $\operatorname{len} \langle f(m), \dots, f(n) \rangle \leq \operatorname{len} f$.

(9) For every directed chain c of G such that $1 \leq m$ and $m \leq n$ and $n \leq \operatorname{len} c$ holds $\langle c(m), \dots, c(n) \rangle$ is a directed chain of G.

(10) For every non empty directed chain o_1 of G holds len vertex-seq(o_1) $= \operatorname{len} o_1 + 1$.

Let us consider G and let o_1 be a directed non empty chain of G. Observe that vertex-seq(o_1) is non empty.

One can prove the following propositions:

(11) Let o_1 be a directed non empty chain of G and given n. Suppose $1 \leq n$ and $n \leq \operatorname{len} o_1$. Then (vertex-seq($o_1$))($n$) = (the source of G)($o_1(n)$) and (vertex-seq(o_1))($n+1$) = (the target of G)($o_1(n)$).

(12) For every non empty finite sequence f such that $1 \leq m$ and $m \leq n$ and $n \leq \operatorname{len} f$ holds $\langle f(m), \ldots, f(n) \rangle$ is non empty.

(13) For all directed chains c, c_1 of G such that $1 \leq m$ and $m \leq n$ and $n \leq \operatorname{len} c$ and $c_1 = \langle c(m), \ldots, c(n) \rangle$ holds vertex-seq(c_1) = $\langle$(vertex-seq(c))(m), ..., (vertex-seq(c))($n+1$)$\rangle$.

(14) For every directed non empty chain o_1 of G holds (vertex-seq(o_1))($\operatorname{len} o_1 + 1$) = (the target of G)($o_1(\operatorname{len} o_1)$).

(15) For all directed non empty chains c_1, c_2 of G holds (vertex-seq(c_1))($\operatorname{len} c_1 + 1$) = (vertex-seq($c_2$))(1) iff $c_1 \frown c_2$ is a directed non empty chain of G.

(16) For all directed non empty chains c, c_1, c_2 of G such that $c = c_1 \frown c_2$ holds (vertex-seq(c))(1) = (vertex-seq(c_1))(1) and (vertex-seq(c))($\operatorname{len} c + 1$) = (vertex-seq(c_2))($\operatorname{len} c_2 + 1$).

(17) For every directed non empty chain o_1 of G such that o_1 is cyclic holds (vertex-seq(o_1))(1) = (vertex-seq(o_1))($\operatorname{len} o_1 + 1$).

(18) Let c be a directed non empty chain of G. Suppose c is cyclic. Given n. Then there exists a directed chain c_3 of G such that $\operatorname{len} c_3 = n$ and $c_3 \frown c$ is a directed non empty chain of G.

Let I_1 be a graph. We say that I_1 is directed cycle-less if and only if:

(Def. 4) For every directed chain d_1 of I_1 such that d_1 is non empty holds d_1 is non cyclic.

We introduce I_1 has directed cycle as an antonym of I_1 is directed cycle-less.

Let us mention that every graph which is empty is also directed cycle-less.

Let I_1 be a graph. We say that I_1 is well-founded if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let v be an element of the vertices of I_1. Then there exists n such that for every directed chain c of I_1 if c is non empty and (vertex-seq(c))($\operatorname{len} c + 1$) = v, then $\operatorname{len} c \leq n$.

Let G be an empty graph. Note that every chain of G is empty.

One can check that every graph which is empty is also well-founded.

Let us observe that every graph which is non well-founded is also non empty.

One can check that there exists a graph which is well-founded.

Let us note that every graph which is well-founded is also directed cycle-less.

Let us note that there exists a graph which is non well-founded.

One can verify that there exists a graph which is directed cycle-less.

We now state the proposition

(19) For every decorated tree t and for every node p of t and for every natural number k holds $p \restriction k$ is a node of t.

2. Some properties of many sorted algebras

Next we state two propositions:

(20) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set indexed by the carrier of S, and let t be a term of S over X. Suppose t is not root. Then there exists an operation symbol o of S such that $t(\varepsilon) = \langle o, \text{the carrier of } S\rangle$.

(21) Let S be a non void non empty many sorted signature, and let A be an algebra over S, and let G be a generator set of A, and let B be a subset of A. If $G \subseteq B$, then B is a generator set of A.

Let S be a non void non empty many sorted signature and let A be a finitely-generated non-empty algebra over S. Note that there exists a generator set of A which is non-empty and locally-finite.

One can prove the following two propositions:

(22) Let S be a non void non empty many sorted signature, and let A be a non-empty algebra over S, and let X be a non-empty generator set of A. Then there exists many sorted function from $\mathrm{Free}(X)$ into A which is an epimorphism of $\mathrm{Free}(X)$ onto A

(23) Let S be a non void non empty many sorted signature, and let A be a non-empty algebra over S, and let X be a non-empty generator set of A. If A is non locally-finite, then $\mathrm{Free}(X)$ is non locally-finite.

Let S be a non void non empty many sorted signature, let X be a non-empty locally-finite many sorted set indexed by the carrier of S, and let v be a sort symbol of S. One can check that $\mathrm{FreeGenerator}(v, X)$ is finite.

One can prove the following propositions:

(24) Let S be a non void non empty many sorted signature, and let X be a non-empty locally-finite many sorted set indexed by the carrier of S, and let v be a sort symbol of S. Then $\mathrm{FreeGenerator}(v, X)$ is finite.

(25) Let S be a non void non empty many sorted signature, and let A be a non-empty algebra over S, and let o be an operation symbol of S. If (the arity of S)$(o) = \varepsilon$, then $\mathrm{dom}\,\mathrm{Den}(o, A) = \{\varepsilon\}$.

Let I_1 be a non void non empty many sorted signature. We say that I_1 is finitely operated if and only if:

(Def. 6) For every sort symbol s of I_1 holds $\{o : o$ ranges over operation symbols of I_1, the result sort of $o = s\}$ is finite.

Next we state three propositions:

(26) Let S be a non void non empty many sorted signature, and let A be a non-empty algebra over S, and let v be a sort symbol of S. If S is finitely operated, then $\mathrm{Constants}(A, v)$ is finite.

(27) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set indexed by the carrier of S, and let v be a sort

symbol of S Then $\{t : t$ ranges over elements of (the sorts of $\mathrm{Free}(X))(v)$, $\mathrm{depth}(t) = 0\} = \mathrm{FreeGenerator}(v, X) \cup \mathrm{Constants}(\mathrm{Free}(X), v)$.

(28) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set indexed by the carrier of S, and let v, v_1 be sort symbols of S, and let o be an operation symbol of S, and let t be an element of (the sorts of $\mathrm{Free}(X))(v)$, and let a be an argument sequence of $\mathrm{Sym}(o, X)$, and let k be a natural number, and let a_1 be an element of (the sorts of $\mathrm{Free}(X))(v_1)$. If $t = \langle o$, the carrier of $S\rangle$-tree(a) and $k \in \mathrm{dom}\, a$ and $a_1 = a(k)$, then $\mathrm{depth}(a_1) < \mathrm{depth}(t)$.

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