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The Correspondence Between Monotonic Many Sorted Signatures and Well-Founded Graphs. Part II [1]

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Summary. The graph induced by a many sorted signature is defined as follows: the vertices are the symbols of sorts, and if a sort s is an argument of an operation with result sort t, then a directed edge $[s, t]$ is in the graph. The key lemma states relationship between the depth of elements of a free many sorted algebra over a signature and the length of directed chains in the graph induced by the signature. Then we prove that a monotonic many sorted signature (every finitely-generated algebra over it is locally-finite) induces a *well-founded* graph. The converse holds with an additional assumption that the signature is finitely operated, i.e. there is only a finite number of operations with the given result sort.

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The articles [30], [33], [19], [2], [15], [31], [34], [12], [14], [13], [18], [21], [17], [10], [3], [5], [7], [1], [4], [26], [6], [32], [20], [22], [29], [28], [11], [27], [25], [24], [23], [8], [9], and [16] provide the terminology and notation for this paper.

In this paper n will be a natural number.

Let S be a many sorted signature. The functor InducedEdges(S) yields a set and is defined by the condition (Def. 1).

(Def. 1) Let x be a set. Then $x \in$ InducedEdges(S) if and only if there exist sets o_1, v such that $x = \langle o_1, v \rangle$ and $o_1 \in$ the operation symbols of S and $v \in$ the carrier of S and there exists a natural number n and there exists an element a_1 of (the carrier of S)* such that (the arity of S)$(o_1) = a_1$ and $n \in \operatorname{dom} a_1$ and $a_1(n) = v$.

Next we state the proposition

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(1) For every many sorted signature S holds InducedEdges(S) $\subseteq$ [: the operation symbols of S, the carrier of S :].

Let S be a many sorted signature. The functor InducedSource(S) yields a function from InducedEdges(S) into the carrier of S and is defined as follows:

(Def. 2) For every set e such that $e \in$ InducedEdges(S) holds (InducedSource(S)) $(e) = e_{\mathbf{2}}$.

The functor InducedTarget(S) yielding a function from InducedEdges(S) into the carrier of S is defined by:

(Def. 3) For every set e such that $e \in$ InducedEdges(S) holds (InducedTarget(S)) $(e) =$ (the result sort of S)$(e_{\mathbf{1}})$.

Let S be a non empty many sorted signature. The functor InducedGraph(S) yields a graph and is defined by:

(Def. 4) InducedGraph(S) = $\langle$the carrier of S, InducedEdges(S), InducedSource(S), InducedTarget(S)$\rangle$.

One can prove the following propositions:

(2) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set indexed by the carrier of S, and let v be a sort symbol of S, and given n. Suppose $1 \leq n$. Then there exists an element t of (the sorts of Free(X))(v) such that depth(t) $= n$ if and only if there exists a directed chain c of InducedGraph(S) such that $\operatorname{len} c = n$ and (vertex-seq(c))($\operatorname{len} c + 1$) $= v$.

(3) For every void non empty many sorted signature S holds S is monotonic iff InducedGraph(S) is well-founded.

(4) For every non void non empty many sorted signature S such that S is monotonic holds InducedGraph(S) is well-founded.

(5) Let S be a non void non empty many sorted signature and let X be a non-empty locally-finite many sorted set indexed by the carrier of S. Suppose S is finitely operated. Let n be a natural number and let v be a sort symbol of S. Then $\{t : t$ ranges over elements of (the sorts of Free(X))(v), depth(t) $\leq n\}$ is finite.

(6) Let S be a non void non empty many sorted signature. If S is finitely operated and InducedGraph(S) is well-founded, then S is monotonic.

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