FORMALIZED MATHEMATICS
Volume 5, Number 4, 1996
Warsaw University - Białystok

Translations, Endomorphisms, and Stable Equational Theories

Grzegorz Bancerek
Institute of Mathematics
Polish Academy of Sciences

Summary. Equational theories of an algebra, i.e. the equivalence relation closed under translations and endomorphisms, are formalized. The correspondence between equational theories and term rewriting systems is discussed in the paper. We get as the main result that any pair of elements of an algebra belongs to the equational theory generated by a set A of axioms iff the elements are convertible w.r.t. term rewriting reduction determined by A.

The theory is developed according to [24].

MML Identifier: `MSUALG_6`.

The papers [20], [23], [9], [10], [1], [21], [25], [26], [17], [11], [3], [6], [7], [4], [8], [2], [22], [14], [19], [15], [18], [12], [13], [16], and [5] provide the terminology and notation for this paper.

1. Endomorphisms and translations

Let S be a non empty many sorted signature, let A be an algebra over S, and let s be a sort symbol of S. An element of A, s is an element of (the sorts of A)(s).

Let I be a set, let A be a many sorted set indexed by I, and let h_1, h_2 be many sorted functions from A into A. Then $h_2 \circ h_1$ is a many sorted function from A into A.

The following two propositions are true:

(1) Let S be a non empty non void many sorted signature, and let A be an algebra over S, and let o be an operation symbol of S, and let a be a set. If $a \in \mathrm{Args}(o, A)$, then a is a function.

ISSN 1426–2630

(2) Let S be a non empty non void many sorted signature, and let A be an algebra over S, and let o be an operation symbol of S, and let a be a function. Suppose $a \in \mathrm{Args}(o, A)$. Then $\mathrm{dom}\, a = \mathrm{dom}\,\mathrm{Arity}(o)$ and for every natural number i such that $i \in \mathrm{dom}\,\mathrm{Arity}(o)$ holds $a(i) \in$ (the sorts of A)($\pi_i\,\mathrm{Arity}(o)$).

Let S be a non empty non void many sorted signature and let A be an algebra over S. We say that A is feasible if and only if:

(Def. 1) For every operation symbol o of S such that $\mathrm{Args}(o, A) \neq \emptyset$ holds $\mathrm{Result}(o, A) \neq \emptyset$.

Next we state the proposition

(3) Let S be a non empty non void many sorted signature, and let o be an operation symbol of S, and let A be an algebra over S. Then $\mathrm{Args}(o, A) \neq \emptyset$ if and only if for every natural number i such that $i \in \mathrm{dom}\,\mathrm{Arity}(o)$ holds (the sorts of A)($\pi_i\,\mathrm{Arity}(o)$) $\neq \emptyset$.

Let S be a non empty non void many sorted signature. One can check that every algebra over S which is non-empty is also feasible.

Let S be a non empty non void many sorted signature. One can check that there exists an algebra over S which is non-empty.

Let S be a non empty non void many sorted signature and let A be an algebra over S. A many sorted function from A into A is called an endomorphism of A if:

(Def. 2) It is a homomorphism of A into A.

In the sequel S is a non empty non void many sorted signature and A is an algebra over S.

Next we state three propositions:

(4) $\mathrm{id}_{(\text{the sorts of } A)}$ is an endomorphism of A.

(5) Let h_1, h_2 be many sorted functions from A into A, and let o be an operation symbol of S, and let a be an element of $\mathrm{Args}(o, A)$. If $a \in \mathrm{Args}(o, A)$, then $h_2\#(h_1\#a) = (h_2 \circ h_1)\#a$.

(6) For all endomorphisms h_1, h_2 of A holds $h_2 \circ h_1$ is an endomorphism of A.

Let S be a non empty non void many sorted signature, let A be an algebra over S, and let h_1, h_2 be endomorphisms of A. Then $h_2 \circ h_1$ is an endomorphism of A.

Let S be a non empty non void many sorted signature. The functor $\mathrm{TranslRel}(S)$ is a binary relation on the carrier of S and is defined by the condition (Def. 3).

(Def. 3) Let s_1, s_2 be sort symbols of S. Then $\langle s_1, s_2 \rangle \in \mathrm{TranslRel}(S)$ if and only if there exists an operation symbol o of S such that the result sort of $o = s_2$ and there exists a natural number i such that $i \in \mathrm{dom}\,\mathrm{Arity}(o)$ and $\pi_i\,\mathrm{Arity}(o) = s_1$.

We now state three propositions:

(7) Let S be a non empty non void many sorted signature, and let o be an operation symbol of S, and let A be an algebra over S, and let a be a function. Suppose $a \in \mathrm{Args}(o, A)$. Let i be a natural number and let x be an element of A, $\pi_i \mathrm{Arity}(o)$. Then $a +\cdot (i, x) \in \mathrm{Args}(o, A)$.

(8) Let A_1, A_2 be algebras over S, and let h be a many sorted function from A_1 into A_2, and let o be an operation symbol of S. Suppose $\mathrm{Args}(o, A_1) \neq \emptyset$ and $\mathrm{Args}(o, A_2) \neq \emptyset$. Let i be a natural number. Suppose $i \in \mathrm{dom}\, \mathrm{Arity}(o)$. Let x be an element of A_1, $\pi_i \mathrm{Arity}(o)$. Then $h(\pi_i \mathrm{Arity}(o))(x) \in (\text{the sorts of } A_2)(\pi_i \mathrm{Arity}(o))$.

(9) Let S be a non empty non void many sorted signature, and let o be an operation symbol of S, and let i be a natural number. Suppose $i \in \mathrm{dom}\, \mathrm{Arity}(o)$. Let A_1, A_2 be algebras over S, and let h be a many sorted function from A_1 into A_2, and let a, b be elements of $\mathrm{Args}(o, A_1)$. Suppose $a \in \mathrm{Args}(o, A_1)$ and $h\#a \in \mathrm{Args}(o, A_2)$. Let f, g_1, g_2 be functions. Suppose $f = a$ and $g_1 = h\#a$ and $g_2 = h\#b$. Let x be an element of A_1, $\pi_i \mathrm{Arity}(o)$. If $b = f +\cdot (i, x)$, then $g_2(i) = h(\pi_i \mathrm{Arity}(o))(x)$ and $h\#b = g_1 +\cdot (i, g_2(i))$.

Let S be a non empty non void many sorted signature, let o be an operation symbol of S, let i be a natural number, let A be an algebra over S, and let a be a function. The functor $o_i^A(a, -)$ yields a function and is defined by the conditions (Def. 4).

(Def. 4) (i) $\mathrm{dom}(o_i^A(a, -)) = (\text{the sorts of } A)(\pi_i \mathrm{Arity}(o))$, and

(ii) for every set x such that $x \in (\text{the sorts of } A)(\pi_i \mathrm{Arity}(o))$ holds $o_i^A(a, -)(x) = (\mathrm{Den}(o, A))(a +\cdot (i, x))$.

One can prove the following proposition

(10) Let S be a non empty non void many sorted signature, and let o be an operation symbol of S, and let i be a natural number. Suppose $i \in \mathrm{dom}\, \mathrm{Arity}(o)$. Let A be a feasible algebra over S and let a be a function. Suppose $a \in \mathrm{Args}(o, A)$. Then $o_i^A(a, -)$ is a function from (the sorts of A)$(\pi_i \mathrm{Arity}(o))$ into (the sorts of A)(the result sort of o).

Let S be a non empty non void many sorted signature, let s_1, s_2 be sort symbols of S, let A be an algebra over S, and let f be a function. We say that f is an elementary translation in A from s_1 into s_2 if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exists an operation symbol o of S such that

(i) the result sort of $o = s_2$, and

(ii) there exists a natural number i such that $i \in \mathrm{dom}\, \mathrm{Arity}(o)$ and $\pi_i \mathrm{Arity}(o) = s_1$ and there exists a function a such that $a \in \mathrm{Args}(o, A)$ and $f = o_i^A(a, -)$.

One can prove the following propositions:

(11) Let S be a non empty non void many sorted signature, and let s_1, s_2 be sort symbols of S, and let A be a feasible algebra over S, and let f be a function. Suppose f is an elementary translation in A from s_1 into s_2.

Then
(i) f is a function from (the sorts of A)(s_1) into (the sorts of A)(s_2),
(ii) (the sorts of A)(s_1) $\neq \emptyset$, and
(iii) (the sorts of A)(s_2) $\neq \emptyset$.

(12) Let S be a non empty non void many sorted signature, and let s_1, s_2 be sort symbols of S, and let A be an algebra over S, and let f be a function. If f is an elementary translation in A from s_1 into s_2, then $\langle s_1, s_2\rangle \in \mathrm{TranslRel}(S)$.

(13) Let S be a non empty non void many sorted signature, and let s_1, s_2 be sort symbols of S, and let A be a non-empty algebra over S. If $\langle s_1, s_2\rangle \in \mathrm{TranslRel}(S)$, then there exists function which is an elementary translation in A from s_1 into s_2.

(14) Let S be a non empty non void many sorted signature, and let A be a feasible algebra over S, and let s_1, s_2 be sort symbols of S. Suppose TranslRel(S) reduces s_1 to s_2. Let q be a reduction sequence w.r.t. TranslRel(S) and let p be a function yielding finite sequence. Suppose that
(i) $\mathrm{len}\, q = \mathrm{len}\, p + 1$,
(ii) $s_1 = q(1)$,
(iii) $s_2 = q(\mathrm{len}\, q)$, and
(iv) for every natural number i and for every function f and for all sort symbols s_1, s_2 of S such that $i \in \mathrm{dom}\, p$ and $f = p(i)$ and $s_1 = q(i)$ and $s_2 = q(i+1)$ holds f is an elementary translation in A from s_1 into s_2.
Then
(v) $\mathrm{compose}_{(\text{the sorts of } A)(s_1)}\, p$ is a function from (the sorts of A)(s_1) into (the sorts of A)(s_2), and
(vi) if $p \neq \emptyset$, then (the sorts of A)(s_1) $\neq \emptyset$ and (the sorts of A)(s_2) $\neq \emptyset$.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let s_1, s_2 be sort symbols of S. Let us assume that TranslRel(S) reduces s_1 to s_2. A function from (the sorts of A)(s_1) into (the sorts of A)(s_2) is called a translation in A from s_1 into s_2 if it satisfies the condition (Def. 6).

(Def. 6) There exists a reduction sequence q w.r.t. TranslRel(S) and there exists a function yielding finite sequence p such that
(i) it $= \mathrm{compose}_{(\text{the sorts of } A)(s_1)}\, p$,
(ii) $\mathrm{len}\, q = \mathrm{len}\, p + 1$,
(iii) $s_1 = q(1)$,
(iv) $s_2 = q(\mathrm{len}\, q)$, and
(v) for every natural number i and for every function f and for all sort symbols s_1, s_2 of S such that $i \in \mathrm{dom}\, p$ and $f = p(i)$ and $s_1 = q(i)$ and $s_2 = q(i+1)$ holds f is an elementary translation in A from s_1 into s_2.

We now state the proposition

(15) Let S be a non empty non void many sorted signature, and let A be a non-empty algebra over S, and let s_1, s_2 be sort symbols of S. Sup-

pose TranslRel(S) reduces s_1 to s_2. Let q be a reduction sequence w.r.t. TranslRel(S) and let p be a function yielding finite sequence. Suppose that

(i) $\operatorname{len} q = \operatorname{len} p + 1$,

(ii) $s_1 = q(1)$,

(iii) $s_2 = q(\operatorname{len} q)$, and

(iv) for every natural number i and for every function f and for all sort symbols s_1, s_2 of S such that $i \in \operatorname{dom} p$ and $f = p(i)$ and $s_1 = q(i)$ and $s_2 = q(i+1)$ holds f is an elementary translation in A from s_1 into s_2.

Then $\text{compose}_{(\text{the sorts of } A)(s_1)}\, p$ is a translation in A from s_1 into s_2.

In the sequel A is a non-empty algebra over S.

The following propositions are true:

(16) For every sort symbol s of S holds $\text{id}_{(\text{the sorts of } A)(s)}$ is a translation in A from s into s

(17) Let s_1, s_2 be sort symbols of S and let f be a function. Suppose f is an elementary translation in A from s_1 into s_2. Then TranslRel(S) reduces s_1 to s_2 and f is a translation in A from s_1 into s_2.

(18) Let s_1, s_2, s_3 be sort symbols of S. Suppose TranslRel(S) reduces s_1 to s_2 and TranslRel(S) reduces s_2 to s_3. Let t_1 be a translation in A from s_1 into s_2 and let t_2 be a translation in A from s_2 into s_3. Then $t_2 \cdot t_1$ is a translation in A from s_1 into s_3.

(19) Let s_1, s_2, s_3 be sort symbols of S. Suppose TranslRel(S) reduces s_1 to s_2. Let t be a translation in A from s_1 into s_2 and let f be a function. Suppose f is an elementary translation in A from s_2 into s_3. Then $f \cdot t$ is a translation in A from s_1 into s_3.

(20) Let s_1, s_2, s_3 be sort symbols of S. Suppose TranslRel(S) reduces s_2 to s_3. Let f be a function. Suppose f is an elementary translation in A from s_1 into s_2. Let t be a translation in A from s_2 into s_3. Then $t \cdot f$ is a translation in A from s_1 into s_3

The scheme *TranslationInd* concerns a non empty non void many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

Let s_1, s_2 be sort symbols of $\mathcal{A}$. Suppose TranslRel($\mathcal{A}$) reduces s_1 to s_2. Let t be a translation in $\mathcal{B}$ from s_1 into s_2. Then $\mathcal{P}[t, s_1, s_2]$

provided the parameters meet the following requirements:

- For every sort symbol s of $\mathcal{A}$ holds $\mathcal{P}[\text{id}_{(\text{the sorts of } \mathcal{B})(s)}, s, s]$,
- Let s_1, s_2, s_3 be sort symbols of $\mathcal{A}$. Suppose TranslRel($\mathcal{A}$) reduces s_1 to s_2. Let t be a translation in $\mathcal{B}$ from s_1 into s_2. Suppose $\mathcal{P}[t, s_1, s_2]$. Let f be a function. If f is an elementary translation in $\mathcal{B}$ from s_2 into s_3, then $\mathcal{P}[f \cdot t, s_1, s_3]$.

The following propositions are true:

(21) Let A_1, A_2 be non-empty algebras over S and let h be a many sorted function from A_1 into A_2. Suppose h is a homomorphism of A_1 into A_2

Let o be an operation symbol of S and let i be a natural number. Suppose $i \in \operatorname{dom} \operatorname{Arity}(o)$. Let a be an element of $\operatorname{Args}(o, A_1)$. Then h(the result sort of $o) \cdot o_i^{A_1}(a, -) = o_i^{A_2}(h\#a, -) \cdot h(\pi_i \operatorname{Arity}(o))$.

(22) Let h be an endomorphism of A, and let o be an operation symbol of S, and let i be a natural number. Suppose $i \in \operatorname{dom} \operatorname{Arity}(o)$. Let a be an element of $\operatorname{Args}(o, A)$. Then h(the result sort of $o) \cdot o_i^{A}(a, -) = o_i^{A}(h\#a, -) \cdot h(\pi_i \operatorname{Arity}(o))$.

(23) Let A_1, A_2 be non-empty algebras over S and let h be a many sorted function from A_1 into A_2. Suppose h is a homomorphism of A_1 into A_2 Let s_1, s_2 be sort symbols of S and let t be a function. Suppose t is an elementary translation in A_1 from s_1 into s_2. Then there exists a function T from (the sorts of $A_2)(s_1)$ into (the sorts of $A_2)(s_2)$ such that T is an elementary translation in A_2 from s_1 into s_2 and $T \cdot h(s_1) = h(s_2) \cdot t$.

(24) Let h be an endomorphism of A, and let s_1, s_2 be sort symbols of S, and let t be a function. Suppose t is an elementary translation in A from s_1 into s_2. Then there exists a function T from (the sorts of $A)(s_1)$ into (the sorts of $A)(s_2)$ such that T is an elementary translation in A from s_1 into s_2 and $T \cdot h(s_1) = h(s_2) \cdot t$.

(25) Let A_1, A_2 be non-empty algebras over S and let h be a many sorted function from A_1 into A_2. Suppose h is a homomorphism of A_1 into A_2 Let s_1, s_2 be sort symbols of S. Suppose TranslRel(S) reduces s_1 to s_2. Let t be a translation in A_1 from s_1 into s_2. Then there exists a translation T in A_2 from s_1 into s_2 such that $T \cdot h(s_1) = h(s_2) \cdot t$.

(26) Let h be an endomorphism of A and let s_1, s_2 be sort symbols of S. Suppose TranslRel(S) reduces s_1 to s_2. Let t be a translation in A from s_1 into s_2. Then there exists a translation T in A from s_1 into s_2 such that $T \cdot h(s_1) = h(s_2) \cdot t$.

2. Compatibility, invariantness, and stability

Let S be a non empty non void many sorted signature, let A be an algebra over S, and let R be a many sorted relation of A. We say that R is compatible if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let o be an operation symbol of S and let a, b be functions. Suppose $a \in \operatorname{Args}(o, A)$ and $b \in \operatorname{Args}(o, A)$ and for every natural number n such that $n \in \operatorname{dom} \operatorname{Arity}(o)$ holds $\langle a(n), b(n) \rangle \in R(\pi_n \operatorname{Arity}(o))$. Then $\langle (\operatorname{Den}(o, A))(a), (\operatorname{Den}(o, A))(b) \rangle \in R$(the result sort of o).

We say that R is invariant if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let s_1, s_2 be sort symbols of S and let t be a function. Suppose t is an elementary translation in A from s_1 into s_2. Let a, b be sets. If $\langle a, b \rangle \in R(s_1)$, then $\langle t(a), t(b) \rangle \in R(s_2)$.

We say that R is stable if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let h be an endomorphism of A, and let s be a sort symbol of S, and let a, b be sets. If $\langle a, b\rangle \in R(s)$, then $\langle h(s)(a), h(s)(b)\rangle \in R(s)$.

The following propositions are true:

(27) Let R be an equivalence many sorted relation of A. Then R is compatible if and only if R is a congruence of A.

(28) Let R be a many sorted relation of A. Then R is invariant if and only if for all sort symbols s_1, s_2 of S such that TranslRel(S) reduces s_1 to s_2 and for every translation f in A from s_1 into s_2 and for all sets a, b such that $\langle a, b\rangle \in R(s_1)$ holds $\langle f(a), f(b)\rangle \in R(s_2)$.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S. Note that every equivalence many sorted relation of A which is invariant is also compatible and every equivalence many sorted relation of A which is compatible is also invariant.

Let X be a non empty set. Note that $\triangle_X$ is non empty.

Now we present two schemes. The scheme *MSRExistence* deals with a non empty set $\mathcal{A}$, a non-empty many sorted set $\mathcal{B}$ indexed by $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

There exists a many sorted relation R of $\mathcal{B}$ such that for every element i of $\mathcal{A}$ and for all elements a, b of $\mathcal{B}(i)$ holds $\langle a, b\rangle \in R(i)$ if and only if $\mathcal{P}[i, a, b]$

for all values of the parameters.

The scheme *MSRLambdaU* deals with a set $\mathcal{A}$, a many sorted set $\mathcal{B}$ indexed by $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

(i) There exists a many sorted relation R of $\mathcal{B}$ such that for every set i such that $i \in \mathcal{A}$ holds $R(i) = \mathcal{F}(i)$, and

(ii) for all many sorted relations R_1, R_2 of $\mathcal{B}$ such that for every set i such that $i \in \mathcal{A}$ holds $R_1(i) = \mathcal{F}(i)$ and for every set i such that $i \in \mathcal{A}$ holds $R_2(i) = \mathcal{F}(i)$ holds $R_1 = R_2$

provided the parameters meet the following requirement:

- For every set i such that $i \in \mathcal{A}$ holds $\mathcal{F}(i)$ is a relation between $\mathcal{B}(i)$ and $\mathcal{B}(i)$.

Let I be a set and let A be a many sorted set indexed by I. The functor $\triangle^I_A$ yielding a many sorted relation of A is defined by:

(Def. 10) For every set i such that $i \in I$ holds $(\triangle^I_A)(i) = \triangle_{A(i)}$.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S. One can verify that every many sorted relation of A which is equivalence is also non-empty.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S. Observe that there exists a many sorted relation of A which is invariant stable and equivalence.

3. Invariant, stable, and invariant stable closure

In the sequel S will denote a non empty non void many sorted signature, A will denote a non-empty algebra over S, and R will denote a many sorted relation of the sorts of A.

The scheme *MSRelCl* concerns a non empty non void many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, many sorted relations $\mathcal{Q}$, $\mathcal{D}$ of $\mathcal{B}$, a unary predicate $\mathcal{Q}$, and a ternary predicate $\mathcal{P}$, and states that:

$\mathcal{Q}[\mathcal{D}]$ and $\mathcal{Q} \subseteq \mathcal{D}$ and for every many sorted relation P of $\mathcal{B}$ such that $\mathcal{Q}[P]$ and $\mathcal{Q} \subseteq P$ holds $\mathcal{D} \subseteq P$

provided the following requirements are met:

- Let R be a many sorted relation of $\mathcal{B}$. Then $\mathcal{Q}[R]$ if and only if for all sort symbols s_1, s_2 of $\mathcal{A}$ and for every function f from (the sorts of $\mathcal{B}$)(s_1) into (the sorts of $\mathcal{B}$)(s_2) such that $\mathcal{P}[f, s_1, s_2]$ and for all sets a, b such that $\langle a, b\rangle \in R(s_1)$ holds $\langle f(a), f(b)\rangle \in R(s_2)$,
- Let s_1, s_2, s_3 be sort symbols of $\mathcal{A}$, and let f_1 be a function from (the sorts of $\mathcal{B}$)(s_1) into (the sorts of $\mathcal{B}$)(s_2), and let f_2 be a function from (the sorts of $\mathcal{B}$)(s_2) into (the sorts of $\mathcal{B}$)(s_3). If $\mathcal{P}[f_1, s_1, s_2]$ and $\mathcal{P}[f_2, s_2, s_3]$, then $\mathcal{P}[f_2 \cdot f_1, s_1, s_3]$,
- For every sort symbol s of $\mathcal{A}$ holds $\mathcal{P}[\mathrm{id}_{(\text{the sorts of } \mathcal{B})(s)}, s, s]$,
- Let s be a sort symbol of $\mathcal{A}$ and let a, b be element of $\mathcal{B}$, s. Then $\langle a, b\rangle \in \mathcal{D}(s)$ if and only if there exists a sort symbol s' of $\mathcal{A}$ and there exists a function f from (the sorts of $\mathcal{B}$)(s') into (the sorts of $\mathcal{B}$)(s) and there exist element x, y of $\mathcal{B}$, s' such that $\mathcal{P}[f, s', s]$ and $\langle x, y\rangle \in \mathcal{Q}(s')$ and $a = f(x)$ and $b = f(y)$.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let R be a many sorted relation of the sorts of A. The functor $\mathrm{InvCl}(R)$ is an invariant many sorted relation of A and is defined as follows:

(Def. 11) $R \subseteq \mathrm{InvCl}(R)$ and for every invariant many sorted relation Q of A such that $R \subseteq Q$ holds $\mathrm{InvCl}(R) \subseteq Q$.

The following propositions are true:

(29) Let R be a many sorted relation of the sorts of A, and let s be a sort symbol of S, and let a, b be element of A, s. Then $\langle a, b\rangle \in (\mathrm{InvCl}(R))(s)$ if and only if there exists a sort symbol s' of S and there exist element x, y of A, s' and there exists a translation t in A from s' into s such that $\mathrm{TranslRel}(S)$ reduces s' to s and $\langle x, y\rangle \in R(s')$ and $a = t(x)$ and $b = t(y)$.

(30) For every stable many sorted relation R of A holds $\mathrm{InvCl}(R)$ is stable.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let R be a many sorted relation of the sorts of A. The functor $\mathrm{StabCl}(R)$ is a stable many sorted relation of A and is defined by:

(Def. 12) $R \subseteq \mathrm{StabCl}(R)$ and for every stable many sorted relation Q of A such that $R \subseteq Q$ holds $\mathrm{StabCl}(R) \subseteq Q$.

We now state two propositions:

(31) Let R be a many sorted relation of the sorts of A, and let s be a sort symbol of S, and let a, b be element of A, s. Then $\langle a, b\rangle \in (\mathrm{StabCl}(R))(s)$ if and only if there exist element x, y of A, s and there exists an endomorphism h of A such that $\langle x, y\rangle \in R(s)$ and $a = h(s)(x)$ and $b = h(s)(y)$.

(32) InvCl(StabCl(R)) is stable.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let R be a many sorted relation of the sorts of A. The functor TRS(R) is an invariant stable many sorted relation of A and is defined by:

(Def. 13) $R \subseteq \mathrm{TRS}(R)$ and for every invariant stable many sorted relation Q of A such that $R \subseteq Q$ holds $\mathrm{TRS}(R) \subseteq Q$.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let R be a non-empty many sorted relation of A. One can check the following observations:

* InvCl(R) is non-empty,
* StabCl(R) is non-empty, and
* TRS(R) is non-empty.

We now state several propositions:

(33) For every invariant many sorted relation R of A holds $\mathrm{InvCl}(R) = R$.

(34) For every stable many sorted relation R of A holds $\mathrm{StabCl}(R) = R$.

(35) For every invariant stable many sorted relation R of A holds $\mathrm{TRS}(R) = R$.

(36) $\mathrm{StabCl}(R) \subseteq \mathrm{TRS}(R)$ and $\mathrm{InvCl}(R) \subseteq \mathrm{TRS}(R)$ and $\mathrm{StabCl}(\mathrm{InvCl}(R)) \subseteq \mathrm{TRS}(R)$.

(37) $\mathrm{InvCl}(\mathrm{StabCl}(R)) = \mathrm{TRS}(R)$.

(38) Let R be a many sorted relation of the sorts of A, and let s be a sort symbol of S, and let a, b be element of A, s. Then $\langle a, b\rangle \in (\mathrm{TRS}(R))(s)$ if and only if there exists a sort symbol s' of S such that TranslRel(S) reduces s' to s and there exist element l, r of A, s' and there exists an endomorphism h of A and there exists a translation t in A from s' into s such that $\langle l, r\rangle \in R(s')$ and $a = t(h(s')(l))$ and $b = t(h(s')(r))$.

4. Equational theory

One can prove the following propositions:

(39) Let A be a set and let R, E be binary relations on A. Suppose that for all sets a, b such that $a \in A$ and $b \in A$ holds $\langle a, b\rangle \in E$ iff a and b are convertible w.r.t. R. Then E is equivalence relation-like.

(40) Let A be a set, and let R be a binary relation on A, and let E be an equivalence relation of A. Suppose $R \subseteq E$. Let a, b be sets. If $a \in A$ and $b \in A$ and a and b are convertible w.r.t. R, then $\langle a, b\rangle \in E$.

(41) Let A be a non empty set, and let R be a binary relation on A, and let a, b be elements of A. Then $\langle a, b\rangle \in \mathrm{EqCl}(R)$ if and only if a and b are convertible w.r.t. R.

(42) Let S be a non empty set, and let A be a non-empty many sorted set indexed by S and let R be a many sorted relation of A, and let s be an element of S, and let a, b be elements of $A(s)$. Then $\langle a, b\rangle \in (\mathrm{EqCl}(R))(s)$ if and only if a and b are convertible w.r.t. $R(s)$.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S. An equational theory of A is a stable invariant equivalence many sorted relation of A. Let R be a many sorted relation of A. The functor $\mathrm{EqCl}(R, A)$ yielding an equivalence many sorted relation of A is defined as follows:

(Def. 14) $\mathrm{EqCl}(R, A) = \mathrm{EqCl}(R)$.

We now state four propositions:

(43) For every many sorted relation R of A holds $R \subseteq \mathrm{EqCl}(R, A)$.

(44) Let R be a many sorted relation of A and let E be an equivalence many sorted relation of A. If $R \subseteq E$, then $\mathrm{EqCl}(R, A) \subseteq E$.

(45) Let R be a stable many sorted relation of A, and let s be a sort symbol of S, and let a, b be element of A, s. Suppose a and b are convertible w.r.t. $R(s)$. Let h be an endomorphism of A. Then $h(s)(a)$ and $h(s)(b)$ are convertible w.r.t. $R(s)$.

(46) For every stable many sorted relation R of A holds $\mathrm{EqCl}(R, A)$ is stable.

Let us consider S, A and let R be a stable many sorted relation of A. Note that $\mathrm{EqCl}(R, A)$ is stable.

We now state two propositions:

(47) Let R be an invariant many sorted relation of A, and let s_1, s_2 be sort symbols of S, and let a, b be element of A, s_1. Suppose a and b are convertible w.r.t. $R(s_1)$. Let t be a function. Suppose t is an elementary translation in A from s_1 into s_2. Then $t(a)$ and $t(b)$ are convertible w.r.t. $R(s_2)$.

(48) For every invariant many sorted relation R of A holds $\mathrm{EqCl}(R, A)$ is invariant.

Let us consider S, A and let R be an invariant many sorted relation of A. One can check that $\mathrm{EqCl}(R, A)$ is invariant.

Next we state three propositions:

(49) Let S be a non empty set, and let A be a non-empty many sorted set indexed by S, and let R, E be many sorted relations of A. Suppose that for every element s of S and for all elements a, b of $A(s)$ holds $\langle a, b\rangle \in E(s)$ iff a and b are convertible w.r.t. $R(s)$. Then E is equivalence.

(50) Let R, E be many sorted relations of A. Suppose that for every sort symbol s of S and for all element a, b of A, s holds $\langle a, b\rangle \in E(s)$ iff a and b are convertible w.r.t. $(\mathrm{TRS}(R))(s)$. Then E is an equational theory of A.

(51) Let S be a non empty set, and let A be a non-empty many sorted set indexed by S and let R be a many sorted relation of A, and let E be an equivalence many sorted relation of A. Suppose $R \subseteq E$. Let s be an element of S and let a, b be elements of $A(s)$. If a and b are convertible w.r.t. $R(s)$, then $\langle a, b\rangle \in E(s)$.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let R be a many sorted relation of the sorts of A. The functor $\mathrm{EqTh}(R)$ is an equational theory of A and is defined by:

(Def. 15) $R \subseteq \mathrm{EqTh}(R)$ and for every equational theory Q of A such that $R \subseteq Q$ holds $\mathrm{EqTh}(R) \subseteq Q$.

Next we state three propositions:

(52) For every many sorted relation R of A holds $\mathrm{EqCl}(R, A) \subseteq \mathrm{EqTh}(R)$ and $\mathrm{InvCl}(R) \subseteq \mathrm{EqTh}(R)$ and $\mathrm{StabCl}(R) \subseteq \mathrm{EqTh}(R)$ and $\mathrm{TRS}(R) \subseteq \mathrm{EqTh}(R)$.

(53) Let R be a many sorted relation of A, and let s be a sort symbol of S, and let a, b be element of A, s. Then $\langle a, b\rangle \in (\mathrm{EqTh}(R))(s)$ if and only if a and b are convertible w.r.t. $(\mathrm{TRS}(R))(s)$.

(54) For every many sorted relation R of A holds $\mathrm{EqTh}(R) = \mathrm{EqCl}(\mathrm{TRS}(R), A)$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[3] Grzegorz Bancerek. Reduction relations. *Formalized Mathematics*, 5(**4**):469–478, 1996.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Formalized Mathematics*, 5(**4**):485–492, 1996.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[9] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[11] Krzysztof Hryniewiecki. Relations of tolerance. *Formalized Mathematics*, 2(**1**):105–109, 1991.
[12] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Formalized Mathematics*, 5(**1**):61–65, 1996.
[13] Małgorzata Korolkiewicz. Many sorted quotient algebra. *Formalized Mathematics*, 5(**1**):79–84, 1996.

[14] Jarosław Kotowicz, Beata Madras, and Małgorzata Korolkiewicz. Basic notation of universal algebra. *Formalized Mathematics*, 3(**2**):251–253, 1992.
[15] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.
[16] Robert Milewski. Lattice of congruences in many sorted algebra. *Formalized Mathematics*, 5(**4**):479–483, 1996.
[17] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.
[18] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.
[19] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[20] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[21] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[22] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[23] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[24] Wolfgang Wechler. *Universal Algebra for Computer Scientists.* Volume 25 of *EATCS Monographs on TCS*, Springer–Verlag, 1992.
[25] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[26] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received February 9, 1996