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More on the Lattice of Congruences in Many Sorted Algebra

Robert Milewski
Warsaw University
Białystok

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The terminology and notation used in this paper have been introduced in the following articles: [25], [27], [11], [19], [28], [29], [3], [8], [22], [9], [10], [12], [7], [4], [26], [5], [20], [30], [1], [2], [24], [13], [21], [16], [23], [15], [17], [14], [6], and [18].

1. More on the Lattice of Equivalence Relations

For simplicity we follow a convention: Y denotes a set, I denotes a non empty set, M denotes a many sorted set indexed by I, x, y are arbitrary, k denotes a natural number, p denotes a finite sequence, S denotes a non void non empty many sorted signature, and A denotes a non-empty algebra over S.

The following proposition is true

(1) For every natural number n and for every finite sequence p holds $1 \leq n$ and $n < \operatorname{len} p$ iff $n \in \operatorname{dom} p$ and $n + 1 \in \operatorname{dom} p$.

The scheme *NonUniqSeqEx* concerns a natural number $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists p such that $\operatorname{dom} p = \operatorname{Seg} \mathcal{A}$ and for every k such that $k \in \operatorname{Seg} \mathcal{A}$ holds $\mathcal{P}[k, p(k)]$

provided the following requirement is met:

- For every k such that $k \in \operatorname{Seg} \mathcal{A}$ there exists x such that $\mathcal{P}[k, x]$.

The following three propositions are true:

(2) Let a, b be elements of the carrier of $\operatorname{EqRelLatt}(Y)$ and let A, B be equivalence relations of Y. If $a = A$ and $b = B$, then $a \sqsubseteq b$ iff $A \subseteq B$.

(3) $\perp_{\operatorname{EqRelLatt}(Y)} = \triangle_Y$.

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(4) $\top_{\mathrm{EqRelLatt}(Y)} = \nabla_Y$.

Let us consider Y. Note that EqRelLatt(Y) is bounded.

Next we state the proposition

(5) EqRelLatt(Y) is complete.

Let us consider Y. One can check that EqRelLatt(Y) is complete.

The following propositions are true:

(6) For every set Y and for every subset X of the carrier of EqRelLatt(Y) holds $\bigcup X$ is a binary relation on Y.

(7) For every set Y and for every subset X of the carrier of EqRelLatt(Y) holds $\bigcup X \subseteq \bigsqcup X$.

(8) Let Y be a set, and let X be a subset of the carrier of EqRelLatt(Y), and let R be a binary relation on Y. If $R = \bigcup X$, then $\bigsqcup X = \mathrm{EqCl}(R)$.

(9) Let Y be a set, and let X be a subset of the carrier of EqRelLatt(Y), and let R be a binary relation. If $R = \bigcup X$, then $R = R^{\smile}$.

(10) Let Y be a set and let X be a subset of the carrier of EqRelLatt(Y). Suppose $x \in Y$ and $y \in Y$. Then $\langle x, y \rangle \in \bigsqcup X$ if and only if there exists a finite sequence f such that $1 \leq \mathrm{len}\, f$ and $x = f(1)$ and $y = f(\mathrm{len}\, f)$ and for every natural number i such that $1 \leq i$ and $i < \mathrm{len}\, f$ holds $\langle f(i), f(i+1) \rangle \in \bigcup X$.

2. Lattice of Congruences in Many Sorted Algebra as Sublattice of Lattice of Many Sorted Equivalence Relations Inherited Sup's and Inf's

The following proposition is true

(11) For every subset B of the carrier of CongrLatt(A) holds $\bigsqcap_{\mathrm{EqRelLatt(the\ sorts\ of\ } A)} B$ is a congruence of A.

Let us consider S, A and let E be an element of the carrier of EqRelLatt(the sorts of A). The functor CongrCl(E) yields a congruence of A and is defined by the condition (Def. 1).

(Def. 1) $\mathrm{CongrCl}(E) = \bigsqcap_{\mathrm{EqRelLatt(the\ sorts\ of\ } A)} \{x : x$ ranges over elements of the carrier of EqRelLatt(the sorts of A), x is a congruence of $A \ \wedge\ E \sqsubseteq x\}$.

Let us consider S, A and let X be a subset of the carrier of EqRelLatt(the sorts of A). The functor CongrCl(X) yields a congruence of A and is defined by the condition (Def. 2).

(Def. 2) $\mathrm{CongrCl}(X) = \bigsqcap_{\mathrm{EqRelLatt(the\ sorts\ of\ } A)} \{x : x$ ranges over elements of the carrier of EqRelLatt(the sorts of A), x is a congruence of $A \ \wedge\ X \sqsubseteq x\}$.

The following propositions are true:

(12) For every element C of the carrier of EqRelLatt(the sorts of A) such that C is a congruence of A holds $\mathrm{CongrCl}(C) = C$.

(13) For every subset X of the carrier of EqRelLatt(the sorts of A) holds $\mathrm{CongrCl}(\bigsqcup_{\mathrm{EqRelLatt(the\ sorts\ of\ }A)} X) = \mathrm{CongrCl}(X)$.

(14) Let B_1, B_2 be subsets of the carrier of CongrLatt(A) and let C_1, C_2 be congruences of A. Suppose $C_1 = \bigsqcup_{\mathrm{EqRelLatt(the\ sorts\ of\ }A)} B_1$ and $C_2 = \bigsqcup_{\mathrm{EqRelLatt(the\ sorts\ of\ }A)} B_2$. Then $C_1 \sqcup C_2 = \bigsqcup_{\mathrm{EqRelLatt(the\ sorts\ of\ }A)}(B_1 \cup B_2)$.

(15) Let X be a subset of the carrier of CongrLatt(A). Then $\bigsqcup_{\mathrm{EqRelLatt(the\ sorts\ of\ }A)} X = \bigsqcup_{\mathrm{EqRelLatt(the\ sorts\ of\ }A)}\{\bigsqcup_{\mathrm{EqRelLatt(the\ sorts\ of\ }A)} X_0 :$ X_0 ranges over subsets of the carrier of EqRelLatt(the sorts of A), X_0 is a finite subset of $X\}$.

(16) Let i be an element of I and let e be an equivalence relation of $M(i)$. Then there exists an equivalence relation E of M such that $E(i) = e$ and for every element j of I such that $j \neq i$ holds $E(j) = \nabla_{M(j)}$.

Let I be a non empty set, let M be a many sorted set indexed by I, let i be an element of I, and let X be a subset of the carrier of EqRelLatt(M). Then $\pi_i X$ is a subset of the carrier of EqRelLatt($M(i)$) and it can be characterized by the condition:

(Def. 3) $x \in \pi_i X$ iff there exists an equivalence relation E_1 of M such that $x = E_1(i)$ and $E_1 \in X$.

We introduce EqRelSet(X, i) as a synonym of $\pi_i X$.

Next we state four propositions:

(17) Let i be an element of the carrier of S, and let X be a subset of the carrier of EqRelLatt(the sorts of A), and let B be an equivalence relation of the sorts of A. If $B = \bigsqcup X$, then $B(i) = \bigsqcup_{\mathrm{EqRelLatt((the\ sorts\ of\ }A)(i))} \mathrm{EqRelSet}(X, i)$.

(18) For every subset X of the carrier of CongrLatt(A) holds $\bigsqcup_{\mathrm{EqRelLatt(the\ sorts\ of\ }A)} X$ is a congruence of A.

(19) CongrLatt(A) is $\bigsqcap$-inheriting.

(20) CongrLatt(A) is $\bigsqcup$-inheriting.

Let us consider S, A. Observe that CongrLatt(A) is $\bigsqcap$-inheriting and $\bigsqcup$-inheriting.

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