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More on Products of Many Sorted Algebras

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Summary. This article is continuation of an article defining products of many sorted algebras [12]. Some properties of notions such as commute, Frege, Args() are shown in this article. Notions of constant of operations in many sorted algebras and projection of products of family of many sorted algebras are defined. There is also introduced the notion of class of family of many sorted algebras. The main theorem states that product of family of many sorted algebras and product of class of family of many sorted algebras are isomorphic.

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The terminology and notation used in this paper have been introduced in the following articles: [20], [22], [14], [23], [7], [8], [16], [9], [17], [6], [15], [4], [2], [1], [3], [19], [18], [10], [12], [13], [24], [21], [11], and [5].

1. Preliminaries

For simplicity we adopt the following convention: I denotes a non empty set, J denotes a many sorted set indexed by I, S denotes a non void non empty many sorted signature, i denotes an element of I, c denotes a set, A denotes an algebra family of I over S, E_1 denotes an equivalence relation of I, U_0, U_1, U_2 denote algebras over S, s denotes a sort symbol of S, o denotes an operation symbol of S, and f denotes a function.

Let I be a set, let us consider S, and let A_1 be an algebra family of I over S. One can verify that $\prod A_1$ is non-empty.

Let I be a non empty set and let E_1 be an equivalence relation of I. Note that $\mathrm{Classes}\, E_1$ is non empty.

Let I be a set. Then id_I is a many sorted set indexed by I.

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Let us consider I, E_1. Note that Classes E_1 has non empty elements.

Let X be a set with non empty elements. Then id_X is a non-empty many sorted set indexed by X.

Next we state several propositions:

(1) For all functions f, F and for every set A such that $f \in \prod F$ holds $f \restriction A \in \prod(F \restriction A)$.

(2) Let A be an algebra family of I over S, and let s be a sort symbol of S, and let a be a non empty subset of I, and let A_2 be an algebra family of a over S. If $A \restriction a = A_2$, then $\mathrm{Carrier}(A_2, s) = \mathrm{Carrier}(A, s) \restriction a$.

(3) Let i be a set, and let I be a non empty set, and let E_1 be an equivalence relation of I, and let c_1, c_2 be elements of Classes E_1. If $i \in c_1$ and $i \in c_2$, then $c_1 = c_2$.

(4) For all sets X, Y and for every function f such that $f \in Y^X$ holds $\mathrm{dom}\, f = X$ and $\mathrm{rng}\, f \subseteq Y$.

(5) Let D be a non empty set, and let F be a many sorted function of D, and let C be a functional non empty set with common domain. Suppose $C = \mathrm{rng}\, F$. Let d be an element of D and let e be a set. If $d \in \mathrm{dom}\, F$ and $e \in \mathrm{DOM}(C)$, then $F(d)(e) = (\mathrm{commute}(F))(e)(d)$.

2. Constants of Many Sorted Algebras

Let us consider S, U_0 and let o be an operation symbol of S. The functor $\mathrm{const}(o, U_0)$ is defined by:

(Def. 1) $\mathrm{const}(o, U_0) = (\mathrm{Den}(o, U_0))(\varepsilon)$.

Next we state four propositions:

(6) If $\mathrm{Arity}(o) = \varepsilon$ and $\mathrm{Result}(o, U_0) \neq \emptyset$, then $\mathrm{const}(o, U_0) \in \mathrm{Result}(o, U_0)$.

(7) Suppose (the sorts of $U_0)(s) \neq \emptyset$. Then $\mathrm{Constants}(U_0, s) = \{\mathrm{const}(o, U_0) : o$ ranges over elements of the operation symbols of S, the result sort of $o = s \wedge \mathrm{Arity}(o) = \varepsilon\}$.

(8) If $\mathrm{Arity}(o) = \varepsilon$, then $(\mathrm{commute}(\mathrm{OPER}(A)))(o) \in ((\bigcup\{\mathrm{Result}(o, A(i')) :$ i' ranges over elements of $I\})^{\{\square\}})^I$.

(9) If $\mathrm{Arity}(o) = \varepsilon$, then $\mathrm{const}(o, \prod A) \in (\bigcup\{\mathrm{Result}(o, A(i')) : i'$ ranges over elements of $I\})^I$.

Let us consider S, I, o, A. Observe that $\mathrm{const}(o, \prod A)$ is relation-like and function-like.

One can prove the following three propositions:

(10) For every operation symbol o of S such that $\mathrm{Arity}(o) = \varepsilon$ holds $(\mathrm{const}(o, \prod A))(i) = \mathrm{const}(o, A(i))$.

(11) If $\mathrm{Arity}(o) = \varepsilon$ and $\mathrm{dom}\, f = I$ and for every element i of I holds $f(i) = \mathrm{const}(o, A(i))$, then $f = \mathrm{const}(o, \prod A)$.

(12) Let e be an element of $\mathrm{Args}(o, U_1)$. Suppose $e = \varepsilon$ and $\mathrm{Arity}(o) = \varepsilon$ and $\mathrm{Args}(o, U_1) \neq \emptyset$ and $\mathrm{Args}(o, U_2) \neq \emptyset$. Let F be a many sorted function from U_1 into U_2. Then $F\#e = \varepsilon$.

3. Properties of Arguments of Operations in Many Sorted Algebras

Next we state a number of propositions:

(13) Let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2, and let x be an element of $\mathrm{Args}(o, U_1)$. Then $x \in \prod(\mathrm{dom}_\kappa (F \cdot \mathrm{Arity}(o))(\kappa))$.

(14) Let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2, and let x be an element of $\mathrm{Args}(o, U_1)$, and let n be a set. If $n \in \mathrm{dom}\,\mathrm{Arity}(o)$, then $(F\#x)(n) = F(\pi_n \mathrm{Arity}(o))(x(n))$.

(15) Let x be an element of $\mathrm{Args}(o, \prod A)$. Then $x \in ((\bigcup\{(\text{the sorts of } A(i'))(s') : i' \text{ ranges over elements of } I,\ s' \text{ ranges over elements of the carrier of } S\})^I)^{\mathrm{dom}\,\mathrm{Arity}(o)}$.

(16) For every element x of $\mathrm{Args}(o, \prod A)$ and for every set n such that $n \in \mathrm{dom}\,\mathrm{Arity}(o)$ holds $x(n) \in \prod \mathrm{Carrier}(A, \pi_n \mathrm{Arity}(o))$.

(17) Let i be an element of I and let n be a set. Suppose $n \in \mathrm{dom}\,\mathrm{Arity}(o)$. Let s be a sort symbol of S. Suppose $s = \mathrm{Arity}(o)(n)$. Let y be an element of $\mathrm{Args}(o, \prod A)$ and let g be a function. If $g = y(n)$, then $g(i) \in (\text{the sorts of } A(i))(s)$.

(18) For every element y of $\mathrm{Args}(o, \prod A)$ such that $\mathrm{Arity}(o) \neq \varepsilon$ holds $\mathrm{commute}(y) \in \prod(\mathrm{dom}_\kappa A(o)(\kappa))$.

(19) For every element y of $\mathrm{Args}(o, \prod A)$ such that $\mathrm{Arity}(o) \neq \varepsilon$ holds $y \in \mathrm{dom}\,\blacksquare\mathrm{commute}(\mathrm{Frege}(A(o)))$.

(20) Given I, S, A, o and let s be a sort symbol of S. Suppose $s =$ the result sort of o. Let x be an element of $\mathrm{Args}(o, \prod A)$. Then $(\mathrm{Den}(o, \prod A))(x) \in \prod \mathrm{Carrier}(A, s)$.

(21) Given I, S, A, i and let o be an operation symbol of S. Suppose $\mathrm{Arity}(o) \neq \varepsilon$. Let U_1 be a non-empty algebra over S, and let x be an element of $\mathrm{Args}(o, \prod A)$, and let F be a many sorted function from $\prod A$ into U_1. Then $(\mathrm{commute}(x))(i)$ is an element of $\mathrm{Args}(o, A(i))$.

(22) Given I, S, A, i, o, and let x be an element of $\mathrm{Args}(o, \prod A)$, and let n be a set. If $n \in \mathrm{dom}\,\mathrm{Arity}(o)$, then for every function f such that $f = x(n)$ holds $(\mathrm{commute}(x))(i)(n) = f(i)$.

(23) Let o be an operation symbol of S. Suppose $\mathrm{Arity}(o) \neq \emptyset$. Let y be an element of $\mathrm{Args}(o, \prod A)$, and let i' be an element of I, and let g be a function. If $g = (\mathrm{Den}(o, \prod A))(y)$, then $g(i') = (\mathrm{Den}(o, A(i')))((\mathrm{commute}(y))(i'))$.

4. The Projection of Family of Many Sorted Algebras

Let f be a function and let x be a set. The functor $\mathrm{proj}(f, x)$ yields a function and is defined as follows:

(Def. 2) $\mathrm{dom}\,\mathrm{proj}(f, x) = \prod f$ and for every function y such that $y \in \mathrm{dom}\,\mathrm{proj}(f, x)$ holds $(\mathrm{proj}(f, x))(y) = y(x)$.

Let us consider I, S, let A be an algebra family of I over S, and let i be an element of I. The functor $\mathrm{proj}(A, i)$ yielding a many sorted function from $\prod A$ into $A(i)$ is defined by:

(Def. 3) For every element s of the carrier of S holds $(\mathrm{proj}(A, i))(s) = \mathrm{proj}(\mathrm{Carrier}(A, s), i)$.

Next we state several propositions:

(24) For every element x of $\mathrm{Args}(o, \prod A)$ such that $\mathrm{Args}(o, \prod A) \neq \varepsilon$ and $\mathrm{Arity}(o) \neq \emptyset$ and for every element i of I holds $\mathrm{proj}(A, i)\#x = (\mathrm{commute}(x))(i)$.

(25) For every element i of I and for every algebra family A of I over S holds $\mathrm{proj}(A, i)$ is a homomorphism of $\prod A$ into $A(i)$.

(26) Let U_1 be a non-empty algebra over S and let F be a many sorted function of I. Suppose that for every element i of I there exists a many sorted function F_1 from U_1 into $A(i)$ such that $F_1 = F(i)$ and F_1 is a homomorphism of U_1 into $A(i)$ Then $F \in (\{F(i')(s_1) : s_1$ ranges over sort symbols of S, i' ranges over elements of $I\}^{\text{the carrier of } S})^I$ and $(\mathrm{commute}(F))(s)(i) = F(i)(s)$.

(27) Let U_1 be a non-empty algebra over S and let F be a many sorted function of I. Suppose that for every element i of I there exists a many sorted function F_1 from U_1 into $A(i)$ such that $F_1 = F(i)$ and F_1 is a homomorphism of U_1 into $A(i)$ Then $(\mathrm{commute}(F))(s) \in ((\bigcup\{(\text{the sorts of } A(i'))(s_1) : i'$ ranges over elements of I, s_1 ranges over sort symbols of $S\})^{(\text{the sorts of } U_1)(s)})^I$.

(28) Let U_1 be a non-empty algebra over S and let F be a many sorted function of I. Suppose that for every element i of I there exists a many sorted function F_1 from U_1 into $A(i)$ such that $F_1 = F(i)$ and F_1 is a homomorphism of U_1 into $A(i)$ Let F' be a many sorted function from U_1 into $A(i)$. Suppose $F' = F(i)$. Let x be a set. Suppose $x \in (\text{the sorts of } U_1)(s)$. Let f be a function. If $f = (\mathrm{commute}((\mathrm{commute}(F))(s)))(x)$, then $f(i) = F'(s)(x)$.

(29) Let U_1 be a non-empty algebra over S and let F be a many sorted function of I. Suppose that for every element i of I there exists a many sorted function F_1 from U_1 into $A(i)$ such that $F_1 = F(i)$ and F_1 is a homomorphism of U_1 into $A(i)$ Let x be a set. If $x \in (\text{the sorts of } U_1)(s)$, then $(\mathrm{commute}((\mathrm{commute}(F))(s)))(x) \in \prod \mathrm{Carrier}(A, s)$.

(30) Let U_1 be a non-empty algebra over S and let F be a many sorted function of I. Suppose that for every element i of I there exists a many sorted function F_1 from U_1 into $A(i)$ such that $F_1 = F(i)$ and F_1 is a homomorphism of U_1 into $A(i)$ Then there exists a many sorted function H from U_1 into $\prod A$ such that H is a homomorphism of U_1 into $\prod A$ and for every element i of I holds $\mathrm{proj}(A, i) \circ H = F(i)$.

5. The Class of Family of Many Sorted Algebras

Let us consider I, J, S. A many sorted set indexed by I is said to be a MSAlgebra-Class of S, J if:

(Def. 4) For every set i such that $i \in I$ holds $\mathrm{it}(i)$ is an algebra family of $J(i)$ over S.

Let us consider I, S, A, E_1. The functor $\frac{A}{E_1}$ yields a MSAlgebra-Class of S, $\mathrm{id}_{\mathrm{Classes}\, E_1}$ and is defined by:

(Def. 5) For every c such that $c \in \mathrm{Classes}\, E_1$ holds $(\frac{A}{E_1})(c) = A \restriction c$.

Let us consider I, S, let J be a non-empty many sorted set indexed by I, and let C be a MSAlgebra-Class of S, J. The functor $\prod C$ yields an algebra family of I over S and is defined by the condition (Def. 6).

(Def. 6) Given i. Suppose $i \in I$. Then there exists a non empty set J_1 and there exists an algebra family C_1 of J_1 over S such that $J_1 = J(i)$ and $C_1 = C(i)$ and $(\prod C)(i) = \prod C_1$.

We now state the proposition

(31) Let A be an algebra family of I over S and let E_1 be an equivalence relation of I. Then $\prod A$ and $\prod \prod (\frac{A}{E_1})$ are isomorphic.

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