FORMALIZED MATHEMATICS
Volume 5, Number 4, 1996
Warsaw University - Białystok

Reduction Relations

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Summary. The goal of the article is to start the formalization of Knuth-Bendix completion method (see [2,11] or [1]; see also [12,10]), i.e. to formalize the concept of the completion of a reduction relation. The completion of a reduction relation R is a complete reduction relation equivalent to R such that convertible elements have the same normal forms. The theory formalized in the article includes concepts and facts concerning normal forms, terminating reductions, Church-Rosser property, and equivalence of reduction relations.

MML Identifier: `REWRITE1`.

The terminology and notation used here are introduced in the following articles: [16], [17], [9], [3], [6], [18], [19], [4], [13], [14], [5], [15], [7], and [8].

1. Forgetting concatenation and reduction sequence

Let p, q be finite sequences. The functor $p \,{}^{\$\frown}\, q$ yielding a finite sequence is defined as follows:

(Def. 1) (i) $p \,{}^{\$\frown}\, q = p \frown q$ if $p = \varepsilon$ or $q = \varepsilon$,

(ii) there exists a natural number i and there exists a finite sequence r such that $\operatorname{len} p = i + 1$ and $r = p \restriction \operatorname{Seg} i$ and $p \,{}^{\$\frown}\, q = r \frown q$, otherwise.

In the sequel p, q are finite sequences and x, y are sets.

We now state several propositions:

(1) $\varepsilon \,{}^{\$\frown}\, p = p$ and $p \,{}^{\$\frown}\, \varepsilon = p$.

(2) If $q \neq \varepsilon$, then $(p \frown \langle x \rangle) \,{}^{\$\frown}\, q = p \frown q$.

(3) $(p \frown \langle x \rangle) \,{}^{\$\frown}\, (\langle y \rangle \frown q) = p \frown \langle y \rangle \frown q$.

(4) If $q \neq \varepsilon$, then $\langle x \rangle \,{}^{\$\frown}\, q = q$.

(5) If $p \neq \varepsilon$, then there exist x, q such that $p = \langle x \rangle \frown q$ and $\operatorname{len} p = \operatorname{len} q + 1$.

ISSN 1426–2630

The scheme *PathCatenation* concerns finite sequences $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

Let i be a natural number. Suppose $i \in \mathrm{dom}(\mathcal{A} \,{}^{\$\frown}\, \mathcal{B})$ and $i+1 \in \mathrm{dom}(\mathcal{A} \,{}^{\$\frown}\, \mathcal{B})$. Let x, y be sets. If $x = (\mathcal{A} \,{}^{\$\frown}\, \mathcal{B})(i)$ and $y = (\mathcal{A} \,{}^{\$\frown}\, \mathcal{B})(i+1)$, then $\mathcal{P}[x, y]$

provided the parameters satisfy the following conditions:

- For every natural number i such that $i \in \mathrm{dom}\,\mathcal{A}$ and $i+1 \in \mathrm{dom}\,\mathcal{A}$ holds $\mathcal{P}[\mathcal{A}(i), \mathcal{A}(i+1)]$,
- For every natural number i such that $i \in \mathrm{dom}\,\mathcal{B}$ and $i+1 \in \mathrm{dom}\,\mathcal{B}$ holds $\mathcal{P}[\mathcal{B}(i), \mathcal{B}(i+1)]$,
- $\mathrm{len}\,\mathcal{A} > 0$ and $\mathrm{len}\,\mathcal{B} > 0$ and $\mathcal{A}(\mathrm{len}\,\mathcal{A}) = \mathcal{B}(1)$.

Let R be a binary relation. A finite sequence is said to be a reduction sequence w.r.t. R if:

(Def. 2) len it > 0 and for every natural number i such that $i \in$ dom it and $i+1 \in$ dom it holds $\langle \mathrm{it}(i), \mathrm{it}(i+1) \rangle \in R$.

Next we state the proposition

(6) For every binary relation R and for every reduction sequence p w.r.t. R holds $1 \in \mathrm{dom}\,p$ and $\mathrm{len}\,p \in \mathrm{dom}\,p$.

Let R be a binary relation. Note that every reduction sequence w.r.t. R is non empty.

One can prove the following propositions:

(7) For every binary relation R and for every set a holds $\langle a \rangle$ is a reduction sequence w.r.t. R.

(8) For every binary relation R and for all sets a, b such that $\langle a, b \rangle \in R$ holds $\langle a, b \rangle$ is a reduction sequence w.r.t. R.

(9) Let R be a binary relation and let p, q be reduction sequences w.r.t. R. If $p(\mathrm{len}\,p) = q(1)$, then $p \,{}^{\$\frown}\, q$ is a reduction sequence w.r.t. R.

(10) Let R be a binary relation and let p be a reduction sequence w.r.t. R. Then $\mathrm{Rev}(p)$ is a reduction sequence w.r.t. $R^{\smile}$.

(11) For all binary relations R, Q such that $R \subseteq Q$ holds every reduction sequence w.r.t. R is a reduction sequence w.r.t. Q.

2. Reducibility, convertibility and normal forms

Let R be a binary relation and let a, b be sets. We say that R reduces a to b if and only if:

(Def. 3) There exists a reduction sequence p w.r.t. R such that $p(1) = a$ and $p(\mathrm{len}\,p) = b$.

Let R be a binary relation and let a, b be sets. We say that a and b are convertible w.r.t. R if and only if:

(Def. 4) $R \cup R^{\smile}$ reduces a to b.

One can prove the following propositions:

(12) Let R be a binary relation and let a, b be sets. Then R reduces a to b if and only if there exists a finite sequence p such that $\operatorname{len} p > 0$ and $p(1) = a$ and $p(\operatorname{len} p) = b$ and for every natural number i such that $i \in \operatorname{dom} p$ and $i + 1 \in \operatorname{dom} p$ holds $\langle p(i), p(i + 1)\rangle \in R$.

(13) For every binary relation R and for every set a holds R reduces a to a.

(14) For all sets a, b such that $\emptyset$ reduces a to b holds $a = b$.

(15) For every binary relation R and for all sets a, b such that R reduces a to b and $a \notin \operatorname{field} R$ holds $a = b$.

(16) For every binary relation R and for all sets a, b such that $\langle a, b\rangle \in R$ holds R reduces a to b.

(17) Let R be a binary relation and let a, b, c be sets. Suppose R reduces a to b and R reduces b to c. Then R reduces a to c.

(18) Let R be a binary relation, and let p be a reduction sequence w.r.t. R, and let i, j be natural numbers. If $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and $i \leq j$, then R reduces $p(i)$ to $p(j)$.

(19) For every binary relation R and for all sets a, b such that R reduces a to b and $a \neq b$ holds $a \in \operatorname{field} R$ and $b \in \operatorname{field} R$.

(20) For every binary relation R and for all sets a, b such that R reduces a to b holds $a \in \operatorname{field} R$ iff $b \in \operatorname{field} R$.

(21) For every binary relation R and for all sets a, b holds R reduces a to b iff $a = b$ or $\langle a, b\rangle \in R^*$.

(22) For every binary relation R and for all sets a, b holds R reduces a to b iff R^* reduces a to b.

(23) Let R, Q be binary relations. Suppose $R \subseteq Q$. Let a, b be sets. If R reduces a to b, then Q reduces a to b.

(24) Let R be a binary relation, and let X be a set, and let a, b be sets. Then R reduces a to b if and only if $R \cup \triangle_X$ reduces a to b.

(25) For every binary relation R and for all sets a, b such that R reduces a to b holds $R^\smile$ reduces b to a.

(26) Let R be a binary relation and let a, b be sets. Suppose R reduces a to b. Then a and b are convertible w.r.t. R and b and a are convertible w.r.t. R.

(27) For every binary relation R and for every set a holds a and a are convertible w.r.t. R.

(28) For all sets a, b such that a and b are convertible w.r.t. $\emptyset$ holds $a = b$.

(29) Let R be a binary relation and let a, b be sets. If a and b are convertible w.r.t. R and $a \notin \operatorname{field} R$, then $a = b$.

(30) For every binary relation R and for all sets a, b such that $\langle a, b\rangle \in R$ holds a and b are convertible w.r.t. R.

(31) Let R be a binary relation and let a, b, c be sets. Suppose a and b are convertible w.r.t. R and b and c are convertible w.r.t. R. Then a and c

are convertible w.r.t. R.

(32) Let R be a binary relation and let a, b be sets. Suppose a and b are convertible w.r.t. R. Then b and a are convertible w.r.t. R.

(33) Let R be a binary relation and let a, b be sets. If a and b are convertible w.r.t. R and $a \neq b$, then $a \in \operatorname{field} R$ and $b \in \operatorname{field} R$.

Let R be a binary relation and let a be a set. We say that a is a normal form w.r.t. R if and only if:

(Def. 5) It is not true that there exists a set b such that $\langle a, b \rangle \in R$.

The following propositions are true:

(34) Let R be a binary relation and let a, b be sets. If a is a normal form w.r.t. R and R reduces a to b, then $a = b$.

(35) For every binary relation R and for every set a such that $a \notin \operatorname{field} R$ holds a is a normal form w.r.t. R.

Let R be a binary relation and let a, b be sets. We say that b is a normal form of a w.r.t. R if and only if:

(Def. 6) b is a normal form w.r.t. R and R reduces a to b.

We say that a and b are convergent w.r.t. R if and only if:

(Def. 7) There exists a set c such that R reduces a to c and R reduces b to c.

We say that a and b are divergent w.r.t. R if and only if:

(Def. 8) There exists a set c such that R reduces c to a and R reduces c to b.

We say that a and b are convergent at most in 1 step w.r.t. R if and only if:

(Def. 9) There exists a set c such that $\langle a, c \rangle \in R$ or $a = c$ but $\langle b, c \rangle \in R$ or $b = c$.

We say that a and b are divergent at most in 1 step w.r.t. R if and only if:

(Def. 10) There exists a set c such that $\langle c, a \rangle \in R$ or $a = c$ but $\langle c, b \rangle \in R$ or $b = c$.

Next we state a number of propositions:

(36) For every binary relation R and for every set a such that $a \notin \operatorname{field} R$ holds a is a normal form of a w.r.t. R.

(37) Let R be a binary relation and let a, b be sets. Suppose R reduces a to b. Then

- (i) a and b are convergent w.r.t. R,
- (ii) a and b are divergent w.r.t. R,
- (iii) b and a are convergent w.r.t. R, and
- (iv) b and a are divergent w.r.t. R.

(38) Let R be a binary relation and let a, b be sets. Suppose a and b are convergent w.r.t. R or a and b are divergent w.r.t. R. Then a and b are convertible w.r.t. R.

(39) Let R be a binary relation and let a be a set. Then a and a are convergent w.r.t. R and a and a are divergent w.r.t. R.

(40) For all sets a, b such that a and b are convergent w.r.t. $\emptyset$ or a and b are divergent w.r.t. $\emptyset$ holds $a = b$.

(41) Let R be a binary relation and let a, b be sets. Suppose a and b are convergent w.r.t. R. Then b and a are convergent w.r.t. R.

(42) Let R be a binary relation and let a, b be sets. Suppose a and b are divergent w.r.t. R. Then b and a are divergent w.r.t. R.

(43) Let R be a binary relation and let a, b, c be sets. Suppose that
(i) R reduces a to b and b and c are convergent w.r.t. R, or
(ii) a and b are convergent w.r.t. R and R reduces c to b.
Then a and c are convergent w.r.t. R.

(44) Let R be a binary relation and let a, b, c be sets. Suppose that
(i) R reduces b to a and b and c are divergent w.r.t. R, or
(ii) a and b are divergent w.r.t. R and R reduces b to c.
Then a and c are divergent w.r.t. R.

(45) Let R be a binary relation and let a, b be sets. Suppose a and b are convergent at most in 1 step w.r.t. R. Then a and b are convergent w.r.t. R.

(46) Let R be a binary relation and let a, b be sets. Suppose a and b are divergent at most in 1 step w.r.t. R. Then a and b are divergent w.r.t. R.

Let R be a binary relation and let a be a set. We say that a has a normal form w.r.t. R if and only if:

(Def. 11) There exists set which is a normal form of a w.r.t. R.

Next we state the proposition

(47) For every binary relation R and for every set a such that $a \notin \operatorname{field} R$ holds a has a normal form w.r.t. R.

Let R be a binary relation and let a be a set. Let us assume that a has a normal form w.r.t. R and for all sets b, c such that b is a normal form of a w.r.t. R and c is a normal form of a w.r.t. R holds $b = c$. The functor $\operatorname{nf}_R(a)$ is defined by:

(Def. 12) $\operatorname{nf}_R(a)$ is a normal form of a w.r.t. R.

3. Terminating reductions

Let R be a binary relation. We say that R is reversely well founded if and only if:

(Def. 13) $R^{\smile}$ is well founded.

We say that R is weakly-normalizing if and only if:

(Def. 14) For every set a such that $a \in \operatorname{field} R$ holds a has a normal form w.r.t. R.

We say that R is strongly-normalizing if and only if:

(Def. 15) For every many sorted set f indexed by $\mathbb{N}$ there exists a natural number i such that $\langle f(i), f(i+1)\rangle \notin R$.

Let R be a binary relation. Let us observe that R is reversely well founded if and only if the condition (Def. 16) is satisfied.

(Def. 16) Let Y be a set. Suppose $Y \subseteq \operatorname{field} R$ and $Y \neq \emptyset$. Then there exists a set a such that $a \in Y$ and for every set b such that $b \in Y$ and $a \neq b$ holds $\langle a, b\rangle \notin R$.

The scheme *coNoetherianInduction* deals with a binary relation $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every set a such that $a \in \operatorname{field} \mathcal{A}$ holds $\mathcal{P}[a]$

provided the parameters meet the following conditions:

- $\mathcal{A}$ is reversely well founded,
- For every set a such that for every set b such that $\langle a, b\rangle \in \mathcal{A}$ and $a \neq b$ holds $\mathcal{P}[b]$ holds $\mathcal{P}[a]$.

One can check that every binary relation which is strongly-normalizing is also irreflexive and reversely well founded and every binary relation which is reversely well founded and irreflexive is also strongly-normalizing.

Let us note that every binary relation which is empty is also weakly-normalizing and strongly-normalizing.

Let us note that there exists a binary relation which is empty.

Next we state the proposition

(48) Let Q be a reversely well founded binary relation and let R be a binary relation. If $R \subseteq Q$, then R is reversely well founded.

Let us observe that every binary relation which is strongly-normalizing is also weakly-normalizing.

4. Church-Rosser property

Let R, Q be binary relations. We say that R commutes-weakly with Q if and only if the condition (Def. 17) is satisfied.

(Def. 17) Let a, b, c be sets. Suppose $\langle a, b\rangle \in R$ and $\langle a, c\rangle \in Q$. Then there exists a set d such that Q reduces b to d and R reduces c to d.

Let us notice that the predicate defined above is symmetric. We say that R commutes with Q if and only if the condition (Def. 18) is satisfied.

(Def. 18) Let a, b, c be sets. Suppose R reduces a to b and Q reduces a to c. Then there exists a set d such that Q reduces b to d and R reduces c to d.

Let us notice that the predicate introduced above is symmetric.

We now state the proposition

(49) For all binary relations R, Q such that R commutes with Q holds R commutes-weakly with Q.

Let R be a binary relation. We say that R has unique normal form property if and only if the condition (Def. 19) is satisfied.

(Def. 19) Let a, b be sets. Suppose a is a normal form w.r.t. R and b is a normal form w.r.t. R and a and b are convertible w.r.t. R. Then $a = b$.

We say that R has normal form property if and only if the condition (Def. 20) is satisfied.

(Def. 20) Let a, b be sets. Suppose a is a normal form w.r.t. R and a and b are convertible w.r.t. R. Then R reduces b to a.

We say that R is subcommutative if and only if:

(Def. 21) For all sets a, b, c such that $\langle a, b\rangle \in R$ and $\langle a, c\rangle \in R$ holds b and c are convergent at most in 1 step w.r.t. R.

We introduce R has diamond property as a synonym of R is subcommutative.
We say that R is confluent if and only if:

(Def. 22) For all sets a, b such that a and b are divergent w.r.t. R holds a and b are convergent w.r.t. R.

We say that R has Church-Rosser property if and only if:

(Def. 23) For all sets a, b such that a and b are convertible w.r.t. R holds a and b are convergent w.r.t. R.

We say that R is locally-confluent if and only if:

(Def. 24) For all sets a, b, c such that $\langle a, b\rangle \in R$ and $\langle a, c\rangle \in R$ holds b and c are convergent w.r.t. R.

We introduce R has weak Church-Rosser property as a synonym of R is locally-confluent.

Next we state four propositions:

(50) Let R be a binary relation. Suppose R is subcommutative. Let a, b, c be sets. Suppose R reduces a to b and $\langle a, c\rangle \in R$. Then b and c are convergent w.r.t. R.

(51) For every binary relation R holds R is confluent iff R commutes with R.

(52) Let R be a binary relation. Then R is confluent if and only if for all sets a, b, c such that R reduces a to b and $\langle a, c\rangle \in R$ holds b and c are convergent w.r.t. R

(53) For every binary relation R holds R is locally-confluent iff R commutes-weakly with R.

One can verify the following observations:

* every binary relation which has Church-Rosser property is confluent,
* every binary relation which is confluent is also locally-confluent and has Church-Rosser property,
* every binary relation which is subcommutative is also confluent,
* every binary relation which has Church-Rosser property has also normal form property,
* every binary relation which has normal form property has also unique normal form property, and

* every binary relation which is weakly-normalizing and has unique normal form property has Church-Rosser property.

One can check that every binary relation which is empty is also subcommutative.

One can verify that there exists a binary relation which is empty.

The following three propositions are true:

(54) Let R be a binary relation with unique normal form property and let a, b, c be sets. Suppose b is a normal form of a w.r.t. R and c is a normal form of a w.r.t. R. Then $b = c$.

(55) Let R be a weakly-normalizing binary relation with unique normal form property and let a be a set. Then $\mathrm{nf}_R(a)$ is a normal form of a w.r.t. R.

(56) Let R be a weakly-normalizing binary relation with unique normal form property and let a, b be sets. If a and b are convertible w.r.t. R, then $\mathrm{nf}_R(a) = \mathrm{nf}_R(b)$.

Let us note that every binary relation which is strongly-normalizing and locally-confluent is also confluent.

Let R be a binary relation. We say that R is complete if and only if:

(Def. 25) R is confluent and strongly-normalizing.

Let us note that every binary relation which is complete is also confluent and strongly-normalizing and every binary relation which is confluent and strongly-normalizing is also complete.

Let us mention that there exists a binary relation which is empty.

Let us note that there exists a non empty binary relation which is complete.

We now state three propositions:

(57) Let R, Q be binary relations with Church-Rosser property. If R commutes with Q, then $R \cup Q$ has Church-Rosser property.

(58) For every binary relation R holds R is confluent iff R^* has weak Church-Rosser property.

(59) For every binary relation R holds R is confluent iff R^* is subcommutative.

5. Completion method

Let R, Q be binary relations. We say that R and Q are equivalent if and only if the condition (Def. 26) is satisfied.

(Def. 26) Let a, b be sets. Then a and b are convertible w.r.t. R if and only if a and b are convertible w.r.t. Q.

Let us observe that the predicate introduced above is symmetric.

Let R be a binary relation and let a, b be sets. We say that a and b are critical w.r.t. R if and only if:

(Def. 27) a and b are divergent at most in 1 step w.r.t. R and a and b are not convergent w.r.t. R.

We now state four propositions:

(60) Let R be a binary relation and let a, b be sets. Suppose a and b are critical w.r.t. R. Then a and b are convertible w.r.t. R.

(61) Let R be a binary relation. Suppose that it is not true that there exist sets a, b such that a and b are critical w.r.t. R Then R is locally-confluent.

(62) Let R, Q be binary relations. Suppose that for all sets a, b such that $\langle a, b\rangle \in Q$ holds a and b are critical w.r.t. R. Then R and $R \cup Q$ are equivalent.

(63) Let R be a binary relation. Then there exists a complete binary relation Q such that

(i) $\operatorname{field} Q \subseteq \operatorname{field} R$, and

(ii) for all sets a, b holds a and b are convertible w.r.t. R iff a and b are convergent w.r.t. Q.

Let R be a binary relation. A complete binary relation is said to be a completion of R if it satisfies the condition (Def. 28).

(Def. 28) Let a, b be sets. Then a and b are convertible w.r.t. R if and only if a and b are convergent w.r.t. it.

Next we state three propositions:

(64) For every binary relation R and for every completion C of R holds R and C are equivalent.

(65) Let R be a binary relation and let Q be a complete binary relation. If R and Q are equivalent, then Q is a completion of R.

(66) Let R be a binary relation, and let C be a completion of R, and let a, b be sets. Then a and b are convertible w.r.t. R if and only if $\mathrm{nf}_C(a) = \mathrm{nf}_C(b)$.

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Received November 14, 1995