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Auxiliary and Approximating Relations[1]

Adam Grabowski
Warsaw University
Białystok

Summary. The aim of this paper is to formalize the second part of Chapter I Section 1 (1.9–1.19) in [12]. Definitions of Scott's auxiliary and approximating relations are introduced in this work. We showed that in a meet-continuous lattice, the way-below relation is the intersection of all approximating auxiliary relations (proposition (40) — compare 1.13 in [12, pp. 43–47]). By (41) a continuous lattice is a complete lattice in which $\ll$ is the smallest approximating auxiliary relation. The notions of the strong interpolation property and the interpolation property are also introduced.

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The articles [21], [25], [19], [10], [23], [24], [20], [9], [3], [26], [28], [7], [8], [27], [2], [4], [22], [18], [1], [17], [13], [29], [14], [15], [5], [11], [16], and [6] provide the notation and terminology for this paper.

1. Auxiliary Relations

Let L be a 1-sorted structure.

(Def. 1) A binary relation on the carrier of L is called a binary relation on L.

Let L be a non empty reflexive relational structure. The functor $\ll_L$ yields a binary relation on L and is defined as follows:

(Def. 2) For all elements x, y of L holds $\langle x, y\rangle \in \ll_L$ iff $x \ll y$.

Let L be a relational structure. The functor $\leqslant_L$ yielding a binary relation on L is defined by:

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(Def. 3) $\leqslant_L$ = the internal relation of L.

Let L be a relational structure and let R be a binary relation on L. We say that R is auxiliary(i) if and only if:

(Def. 4) For all elements x, y of L such that $\langle x, y\rangle \in R$ holds $x \leqslant y$.

We say that R is auxiliary(ii) if and only if:

(Def. 5) For all elements x, y, z, u of L such that $u \leqslant x$ and $\langle x, y\rangle \in R$ and $y \leqslant z$ holds $\langle u, z\rangle \in R$.

Let L be a non empty relational structure and let R be a binary relation on L. We say that R is auxiliary(iii) if and only if:

(Def. 6) For all elements x, y, z of L such that $\langle x, z\rangle \in R$ and $\langle y, z\rangle \in R$ holds $\langle x \sqcup y, z\rangle \in R$.

We say that R is auxiliary(iv) if and only if:

(Def. 7) For every element x of L holds $\langle \perp_L, x\rangle \in R$.

Let L be a non empty relational structure and let R be a binary relation on L. We say that R is auxiliary if and only if:

(Def. 8) R is auxiliary(i), auxiliary(ii), auxiliary(iii), and auxiliary(iv).

Let L be a non empty relational structure. Note that every binary relation on L which is auxiliary is also auxiliary(i), auxiliary(ii), auxiliary(iii), and auxiliary(iv) and every binary relation on L which is auxiliary(i), auxiliary(ii), auxiliary(iii), and auxiliary(iv) is also auxiliary.

Let L be a lower-bounded transitive antisymmetric relational structure with l.u.b.'s. Note that there exists a binary relation on L which is auxiliary.

Next we state the proposition

(1) Let L be a lower-bounded sup-semilattice, A_1 be an auxiliary binary relation on L, and x, y, z, u be elements of L. If $\langle x, z\rangle \in A_1$ and $\langle y, u\rangle \in A_1$, then $\langle x \sqcup y, z \sqcup u\rangle \in A_1$.

Let L be a lower-bounded sup-semilattice. Observe that every binary relation on L which is auxiliary is also transitive.

Let L be a relational structure. Note that $\leqslant_L$ is auxiliary(i).

Let L be a transitive relational structure. One can verify that $\leqslant_L$ is auxiliary(ii).

Let L be an antisymmetric relational structure with l.u.b.'s. One can check that $\leqslant_L$ is auxiliary(iii).

Let L be a lower-bounded antisymmetric non empty relational structure. Note that $\leqslant_L$ is auxiliary(iv).

In the sequel a will denote a set.

Let L be a lower-bounded sup-semilattice. The functor $\mathrm{Aux}(L)$ is defined as follows:

(Def. 9) $a \in \mathrm{Aux}(L)$ iff a is an auxiliary binary relation on L.

Let L be a lower-bounded sup-semilattice. Note that $\mathrm{Aux}(L)$ is non empty.

The following two propositions are true:

(2) For every lower-bounded sup-semilattice L and for every auxiliary binary relation A_1 on L holds $A_1 \subseteq \leqslant_L$.

(3) For every lower-bounded sup-semilattice L holds $\top_{\langle \mathrm{Aux}(L), \subseteq \rangle} = \leqslant_L$.

Let L be a lower-bounded sup-semilattice. Note that $\langle \mathrm{Aux}(L), \subseteq \rangle$ is upper-bounded.

Let L be a non empty relational structure. The functor $\mathrm{AuxBottom}(L)$ yields a binary relation on L and is defined as follows:

(Def. 10) For all elements x, y of L holds $\langle x, y \rangle \in \mathrm{AuxBottom}(L)$ iff $x = \bot_L$.

Let L be a lower-bounded sup-semilattice. Observe that $\mathrm{AuxBottom}(L)$ is auxiliary.

The following propositions are true:

(4) For every lower-bounded sup-semilattice L and for every auxiliary binary relation A_1 on L holds $\mathrm{AuxBottom}(L) \subseteq A_1$.

(5) For every lower-bounded sup-semilattice L and for every auxiliary binary relation A_1 on L holds $\bot_{\langle \mathrm{Aux}(L), \subseteq \rangle} = \mathrm{AuxBottom}(L)$.

Let L be a lower-bounded sup-semilattice. One can verify that $\langle \mathrm{Aux}(L), \subseteq \rangle$ is lower-bounded.

The following two propositions are true:

(6) Let L be a lower-bounded sup-semilattice and a, b be auxiliary binary relations on L. Then $a \cap b$ is an auxiliary binary relation on L.

(7) Let L be a lower-bounded sup-semilattice and X be a non empty subset of $\langle \mathrm{Aux}(L), \subseteq \rangle$. Then $\bigcap X$ is an auxiliary binary relation on L.

Let L be a lower-bounded sup-semilattice. Note that $\langle \mathrm{Aux}(L), \subseteq \rangle$ has g.l.b.'s.

Let L be a lower-bounded sup-semilattice. Observe that $\langle \mathrm{Aux}(L), \subseteq \rangle$ is complete.

Let L be a non empty relational structure, let x be an element of L, and let A_1 be a binary relation on L. The functor $\downarrow_{A_1} x$ yields a subset of L and is defined by:

(Def. 11) $\downarrow_{A_1} x = \{y, y$ ranges over elements of L: $\langle y, x \rangle \in A_1\}$.

The functor $\uparrow_{A_1} x$ yielding a subset of L is defined by:

(Def. 12) $\uparrow_{A_1} x = \{y, y$ ranges over elements of L: $\langle x, y \rangle \in A_1\}$.

One can prove the following proposition

(8) Let L be a lower-bounded sup-semilattice, x be an element of L, and A_1 be an auxiliary(i) binary relation on L. Then $\downarrow_{A_1} x \subseteq \downarrow x$.

Let L be a lower-bounded sup-semilattice, let x be an element of L, and let A_1 be an auxiliary(ii) auxiliary(iii) auxiliary(iv) binary relation on L. Observe that $\downarrow_{A_1} x$ is directed lower and non empty.

Let L be a lower-bounded sup-semilattice and let A_1 be an auxiliary(ii) auxiliary(iii) auxiliary(iv) binary relation on L. The functor $\downarrow A_1$ yields a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ and is defined by:

(Def. 13) For every element x of L holds $(\downarrow A_1)(x) = \downarrow_{A_1} x$.

We now state three propositions:

(9) Let L be a non empty relational structure, A_1 be a binary relation on L, a be a set, and y be an element of L. Then $a \in \twoheaddownarrow_{A_1} y$ if and only if $\langle a, y\rangle \in A_1$.

(10) Let L be a sup-semilattice, A_1 be a binary relation on L, and y be an element of L. Then $a \in \twoheaduparrow_{A_1} y$ if and only if $\langle y, a\rangle \in A_1$.

(11) Let L be a lower-bounded sup-semilattice, A_1 be an auxiliary binary relation on L, and x be an element of L. If $A_1 =$ the internal relation of L, then $\twoheaddownarrow_{A_1} x = \downarrow x$.

Let L be a non empty poset. The functor MonSet(L) yields a strict relational structure and is defined by the conditions (Def. 14).

(Def. 14)(i) $a \in$ the carrier of MonSet(L) iff there exists a map s from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ such that $a = s$ and s is monotone and for every element x of L holds $s(x) \subseteq \downarrow x$, and

(ii) for all sets c, d holds $\langle c, d\rangle \in$ the internal relation of MonSet(L) iff there exist maps f, g from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ such that $c = f$ and $d = g$ and $c \in$ the carrier of MonSet(L) and $d \in$ the carrier of MonSet(L) and $f \leqslant g$.

One can prove the following propositions:

(12) Let L be a lower-bounded sup-semilattice. Then MonSet(L) is a full relational substructure of $(\langle \mathrm{Ids}(L), \subseteq \rangle)^{\text{the carrier of } L}$.

(13) Let L be a lower-bounded sup-semilattice, A_1 be an auxiliary binary relation on L, and x, y be elements of L. If $x \leqslant y$, then $\twoheaddownarrow_{A_1} x \subseteq \twoheaddownarrow_{A_1} y$.

Let L be a lower-bounded sup-semilattice and let A_1 be an auxiliary binary relation on L. Note that $\twoheaddownarrow A_1$ is monotone.

Next we state the proposition

(14) Let L be a lower-bounded sup-semilattice and A_1 be an auxiliary binary relation on L. Then $\twoheaddownarrow A_1 \in$ the carrier of MonSet(L).

Let L be a lower-bounded sup-semilattice. Observe that MonSet(L) is non empty.

Next we state several propositions:

(15) For every lower-bounded sup-semilattice L holds IdsMap(L) $\in$ the carrier of MonSet(L).

(16) For every lower-bounded sup-semilattice L and for every auxiliary binary relation A_1 on L holds $\twoheaddownarrow A_1 \leqslant$ IdsMap(L).

(17) For every lower-bounded non empty poset L and for every ideal I of L holds $\bot_L \in I$.

(18) For every upper-bounded non empty poset L and for every filter F of L holds $\top_L \in F$.

(19) For every lower-bounded non empty poset L holds $\downarrow(\bot_L) = \{\bot_L\}$.

(20) For every upper-bounded non empty poset L holds $\uparrow(\top_L) = \{\top_L\}$.

In the sequel L is a lower-bounded sup-semilattice, A_1 is an auxiliary binary relation on L, and x is an element of L.

The following propositions are true:

(21) The carrier of $L \longmapsto \{\bot_L\}$ is a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$.

(22) The carrier of $L \longmapsto \{\bot_L\} \in$ the carrier of MonSet(L).

(23) $\langle$the carrier of $L \longmapsto \{\bot_L\}$, $\downarrow A_1\rangle \in$ the internal relation of MonSet(L).

Let us consider L. Note that MonSet(L) is upper-bounded.

Let us consider L. The functor Rel2Map(L) yields a map from $\langle \mathrm{Aux}(L), \subseteq \rangle$ into MonSet(L) and is defined by:

(Def. 15) For every A_1 holds $(\mathrm{Rel2Map}(L))(A_1) = \downarrow A_1$.

The following propositions are true:

(24) For all auxiliary binary relations R_1, R_2 on L such that $R_1 \subseteq R_2$ holds $\downarrow R_1 \leqslant \downarrow R_2$.

(25) For all auxiliary binary relations R_1, R_2 on L such that $R_1 \subseteq R_2$ holds $\downarrow_{R_1} x \subseteq \downarrow_{R_2} x$.

Let us consider L. One can verify that Rel2Map(L) is monotone.

Let us consider L. The functor Map2Rel(L) yields a map from MonSet(L) into $\langle \mathrm{Aux}(L), \subseteq \rangle$ and is defined by the condition (Def. 16).

(Def. 16) Let s be a set. Suppose $s \in$ the carrier of MonSet(L). Then there exists an auxiliary binary relation A_1 on L such that

- (i) $A_1 = (\mathrm{Map2Rel}(L))(s)$, and
- (ii) for all sets x, y holds $\langle x, y\rangle \in A_1$ iff there exist elements x', y' of L and there exists a map s' from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ such that $x' = x$ and $y' = y$ and $s' = s$ and $x' \in s'(y')$.

Let us consider L. One can check that Map2Rel(L) is monotone.

We now state two propositions:

(26) $\mathrm{Map2Rel}(L) \cdot \mathrm{Rel2Map}(L) = \mathrm{id}_{\mathrm{dom}\,\mathrm{Rel2Map}(L)}$.

(27) $\mathrm{Rel2Map}(L) \cdot \mathrm{Map2Rel}(L) = \mathrm{id}_{\text{the carrier of MonSet}(L)}$.

Let us consider L. Observe that Rel2Map(L) is one-to-one.

The following three propositions are true:

(28) $(\mathrm{Rel2Map}(L))^{-1} = \mathrm{Map2Rel}(L)$.

(29) Rel2Map(L) is isomorphic.

(30) For every complete lattice L and for every element x of L holds $\bigcap\{I, I$ ranges over ideals of L: $x \leqslant \sup I\} = \downarrow x$.

The scheme *LambdaC'* concerns a non empty relational structure $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

> There exists a function f such that dom $f =$ the carrier of $\mathcal{A}$ and for every element x of $\mathcal{A}$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ and if not $\mathcal{P}[x]$, then $f(x) = \mathcal{G}(x)$

for all values of the parameters.

Let L be a semilattice and let I be an ideal of L. The functor DownMap(I) yields a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ and is defined by:

(Def. 17) For every element x of L holds if $x \leqslant \sup I$, then $(\mathrm{DownMap}(I))(x) = {\downarrow}x \cap I$ and if $x \not\leqslant \sup I$, then $(\mathrm{DownMap}(I))(x) = {\downarrow}x$.

One can prove the following two propositions:

(31) For every semilattice L and for every ideal I of L holds $\mathrm{DownMap}(I) \in$ the carrier of $\mathrm{MonSet}(L)$.

(32) Let L be an antisymmetric reflexive relational structure with g.l.b.'s, x be an element of L, and D be a non empty lower subset of L. Then $\{x\} \sqcap D = {\downarrow}x \cap D$.

2. Approximating Relations

Let L be a non empty relational structure and let A_1 be a binary relation on L. We say that A_1 is approximating if and only if:

(Def. 18) For every element x of L holds $x = \sup {\downarrow}_{A_1} x$.

Let L be a non empty poset and let m_1 be a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$. We say that m_1 is approximating if and only if:

(Def. 19) For every element x of L there exists a subset i_1 of L such that $i_1 = m_1(x)$ and $x = \sup i_1$.

Next we state two propositions:

(33) For every lower-bounded meet-continuous semilattice L and for every ideal I of L holds $\mathrm{DownMap}(I)$ is approximating.

(34) Every lower-bounded continuous lattice is meet-continuous.

Let us mention that every lower-bounded lattice which is continuous is also meet-continuous.

The following proposition is true

(35) For every lower-bounded continuous lattice L and for every ideal I of L holds $\mathrm{DownMap}(I)$ is approximating.

Let L be a non empty reflexive antisymmetric relational structure. Observe that $\ll_L$ is auxiliary(i).

Let L be a non empty reflexive transitive relational structure. One can check that $\ll_L$ is auxiliary(ii).

Let L be a poset with l.u.b.'s. One can verify that $\ll_L$ is auxiliary(iii).

Let L be an inf-complete non empty poset. Note that $\ll_L$ is auxiliary(iii).

Let L be a lower-bounded antisymmetric reflexive non empty relational structure. Observe that $\ll_L$ is auxiliary(iv).

Next we state two propositions:

(36) For every complete lattice L and for every element x of L holds ${\downarrow}_{\ll_L} x = {\downarrow}x$.

(37) For every lattice L holds $\leqslant_L$ is approximating.

Let L be a lower-bounded continuous lattice. One can verify that $\ll_L$ is approximating.

Let L be a complete lattice. Observe that there exists an auxiliary binary relation on L which is approximating.

Let L be a complete lattice. The functor $\mathrm{App}(L)$ is defined as follows:

(Def. 20) $a \in \mathrm{App}(L)$ iff a is an approximating auxiliary binary relation on L.

Let L be a complete lattice. Note that $\mathrm{App}(L)$ is non empty.

Next we state three propositions:

(38) Let L be a complete lattice and m_1 be a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$. Suppose m_1 is approximating and $m_1 \in$ the carrier of $\mathrm{MonSet}(L)$. Then there exists an approximating auxiliary binary relation A_1 on L such that $A_1 = (\mathrm{Map2Rel}(L))(m_1)$.

(39) For every complete lattice L and for every element x of L holds $\bigcap\{(\mathrm{DownMap}(I))(x) : I$ ranges over ideals of $L\} = \downarrow x$.

(40) Let L be a lower-bounded meet-continuous lattice and x be an element of L. Then $\bigcap\{\downarrow_{A_1} x, A_1$ ranges over auxiliary binary relations on L: $A_1 \in \mathrm{App}(L)\} = \downarrow x$.

In the sequel L denotes a complete lattice.

Next we state two propositions:

(41) L is continuous if and only if for every approximating auxiliary binary relation R on L holds $\ll_L \subseteq R$ and $\ll_L$ is approximating.

(42) L is continuous if and only if the following conditions are satisfied:

(i) L is meet-continuous, and

(ii) there exists an approximating auxiliary binary relation R on L such that for every approximating auxiliary binary relation R' on L holds $R \subseteq R'$.

Let L be a non empty relational structure and let A_1 be a binary relation on L. We say that A_1 satisfies strong interpolation property if and only if:

(Def. 21) For all elements x, z of L such that $\langle x, z \rangle \in A_1$ and $x \neq z$ there exists an element y of L such that $\langle x, y \rangle \in A_1$ and $\langle y, z \rangle \in A_1$ and $x \neq y$.

Let L be a non empty relational structure and let A_1 be a binary relation on L. We say that A_1 satisfies interpolation property if and only if:

(Def. 22) For all elements x, z of L such that $\langle x, z \rangle \in A_1$ there exists an element y of L such that $\langle x, y \rangle \in A_1$ and $\langle y, z \rangle \in A_1$.

Next we state two propositions:

(43) Let L be a non empty relational structure, A_1 be a binary relation on L, and x, z be elements of L. If $\langle x, z \rangle \in A_1$ and $x = z$, then there exists an element y of L such that $\langle x, y \rangle \in A_1$ and $\langle y, z \rangle \in A_1$.

(44) Let L be a non empty relational structure and A_1 be a binary relation on L. Suppose A_1 satisfies strong interpolation property. Then A_1 satisfies interpolation property.

Let L be a non empty relational structure. Observe that every binary relation on L which satisfies strong interpolation property satisfies also interpolation property.

In the sequel A_1 is an auxiliary binary relation on L and x, y, z are elements of L.

The following four propositions are true:

(45) Let A_1 be an approximating auxiliary binary relation on L. If $x \nleqslant y$, then there exists an element u of L such that $\langle u, x\rangle \in A_1$ and $u \nleqslant y$.

(46) Let R be an approximating auxiliary binary relation on L. If $\langle x, z\rangle \in R$ and $x \neq z$, then there exists y such that $x \leqslant y$ and $\langle y, z\rangle \in R$ and $x \neq y$.

(47) Let R be an approximating auxiliary binary relation on L. Suppose $x \ll z$ and $x \neq z$. Then there exists an element y of L such that $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$ and $x \neq y$.

(48) For every lower-bounded continuous lattice L holds $\ll_L$ satisfies strong interpolation property.

Let L be a lower-bounded continuous lattice. Observe that $\ll_L$ satisfies strong interpolation property.

Next we state two propositions:

(49) Let L be a lower-bounded continuous lattice and x, y be elements of L. If $x \ll y$, then there exists an element x' of L such that $x \ll x'$ and $x' \ll y$.

(50) Let L be a lower-bounded continuous lattice and x, y be elements of L. Then $x \ll y$ if and only if for every non empty directed subset D of L such that $y \leqslant \sup D$ there exists an element d of L such that $d \in D$ and $x \ll d$.

3. Exercises

Let L be a relational structure, let X be a subset of L, and let R be a binary relation on the carrier of L. We say that X is directed w.r.t. R if and only if:

(Def. 23) For all elements x, y of L such that $x \in X$ and $y \in X$ there exists an element z of L such that $z \in X$ and $\langle x, z\rangle \in R$ and $\langle y, z\rangle \in R$.

We now state the proposition

(51) Let L be a relational structure and X be a subset of L. Suppose X is directed w.r.t. the internal relation of L. Then X is directed.

Let L be a relational structure, let X be a set, let x be an element of L, and let R be a binary relation on the carrier of L. We say that x is maximal w.r.t. X, R if and only if:

(Def. 24) $x \in X$ and it is not true that there exists an element y of L such that $y \in X$ and $y \neq x$ and $\langle x, y\rangle \in R$.

Let L be a relational structure, let X be a set, and let x be an element of L. We say that x is maximal in X if and only if:

(Def. 25) x is maximal w.r.t. X, the internal relation of L.

One can prove the following proposition

(52) Let L be a relational structure, X be a set, and x be an element of L. Then x is maximal in X if and only if the following conditions are satisfied:

(i) $x \in X$, and

(ii) it is not true that there exists an element y of L such that $y \in X$ and $x < y$.

Let L be a relational structure, let X be a set, let x be an element of L, and let R be a binary relation on the carrier of L. We say that x is minimal w.r.t. X, R if and only if:

(Def. 26) $x \in X$ and it is not true that there exists an element y of L such that $y \in X$ and $y \neq x$ and $\langle y, x\rangle \in R$.

Let L be a relational structure, let X be a set, and let x be an element of L. We say that x is minimal in X if and only if:

(Def. 27) x is minimal w.r.t. X, the internal relation of L.

We now state several propositions:

(53) Let L be a relational structure, X be a set, and x be an element of L. Then x is minimal in X if and only if the following conditions are satisfied:

(i) $x \in X$, and

(ii) it is not true that there exists an element y of L such that $y \in X$ and $x > y$.

(54) If A_1 satisfies strong interpolation property, then for every x such that there exists y which is maximal w.r.t. $\downarrow_{A_1} x$, A_1 holds $\langle x, x\rangle \in A_1$.

(55) If for every x such that there exists y which is maximal w.r.t. $\downarrow_{A_1} x$, A_1 holds $\langle x, x\rangle \in A_1$, then A_1 satisfies strong interpolation property.

(56) If A_1 satisfies interpolation property, then for every x holds $\downarrow_{A_1} x$ is directed w.r.t. A_1.

(57) If for every x holds $\downarrow_{A_1} x$ is directed w.r.t. A_1, then A_1 satisfies interpolation property.

(58) Let R be an approximating auxiliary binary relation on L. Suppose R satisfies interpolation property. Then R satisfies strong interpolation property.

Let us consider L. One can verify that every approximating auxiliary binary relation on L which satisfies interpolation property satisfies also strong interpolation property.

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2's Complement Circuit

Katsumi Wasaki
National College of Technology
Nagano

Pauline N. Kawamoto
Shinshu University
Nagano

Summary. This article introduces various Boolean operators which are used in discussing the properties and stability of a 2's complement circuit. We present the definitions and related theorems for the following logical operators which include negative input/output: 'and2a', 'or2a', 'xor2a' and 'nand2a', 'nor2a', etc. We formalize the concept of a 2's complement circuit, define the structures of complementors/incrementors for binary operations, and prove the stability of the circuit.

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The terminology and notation used here are introduced in the following articles: [13], [15], [12], [1], [17], [5], [6], [16], [2], [4], [11], [14], [10], [8], [9], [7], and [3].

1. Boolean Operators

Let x be a set. Then $\langle x \rangle$ is a finite sequence with length 1. Let y be a set. Then $\langle x, y \rangle$ is a finite sequence with length 2. Let z be a set. Then $\langle x, y, z \rangle$ is a finite sequence with length 3.

Let n, m be natural numbers, let p be a finite sequence with length n, and let q be a finite sequence with length m. Then $p \frown q$ is a finite sequence with length $n + m$.

Let S be an unsplit non void non empty many sorted signature, let A be a Boolean circuit of S, let s be a state of A, and let v be a vertex of S. Then $s(v)$ is an element of *Boolean*.

Next we state two propositions:

(1) For every function f and for all sets x_1, x_2 such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ holds $f \cdot \langle x_1, x_2 \rangle = \langle f(x_1), f(x_2) \rangle$.

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(2) For every function f and for all sets x_1, x_2, x_3 such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $x_3 \in \operatorname{dom} f$ holds $f \cdot \langle x_1, x_2, x_3 \rangle = \langle f(x_1), f(x_2), f(x_3) \rangle$.

The function and_2 from *Boolean*^{2} into *Boolean* is defined by:

(Def. 1) For all elements x, y of *Boolean* holds $\mathrm{and}_2(\langle x, y \rangle) = x \wedge y$.

The function and_{2a} from *Boolean*^{2} into *Boolean* is defined by:

(Def. 2) For all elements x, y of *Boolean* holds $(\mathrm{and}_{2a})(\langle x, y \rangle) = \neg x \wedge y$.

The function and_{2b} from *Boolean*^{2} into *Boolean* is defined as follows:

(Def. 3) For all elements x, y of *Boolean* holds $(\mathrm{and}_{2b})(\langle x, y \rangle) = \neg x \wedge \neg y$.

The function nand_2 from *Boolean*^{2} into *Boolean* is defined by:

(Def. 4) For all elements x, y of *Boolean* holds $\mathrm{nand}_2(\langle x, y \rangle) = \neg(x \wedge y)$.

The function nand_{2a} from *Boolean*^{2} into *Boolean* is defined as follows:

(Def. 5) For all elements x, y of *Boolean* holds $(\mathrm{nand}_{2a})(\langle x, y \rangle) = \neg(\neg x \wedge y)$.

The function nand_{2b} from *Boolean*^{2} into *Boolean* is defined as follows:

(Def. 6) For all elements x, y of *Boolean* holds $(\mathrm{nand}_{2b})(\langle x, y \rangle) = \neg(\neg x \wedge \neg y)$.

The function or_2 from *Boolean*^{2} into *Boolean* is defined by:

(Def. 7) For all elements x, y of *Boolean* holds $\mathrm{or}_2(\langle x, y \rangle) = x \vee y$.

The function or_{2a} from *Boolean*^{2} into *Boolean* is defined as follows:

(Def. 8) For all elements x, y of *Boolean* holds $(\mathrm{or}_{2a})(\langle x, y \rangle) = \neg x \vee y$.

The function or_{2b} from *Boolean*^{2} into *Boolean* is defined as follows:

(Def. 9) For all elements x, y of *Boolean* holds $(\mathrm{or}_{2b})(\langle x, y \rangle) = \neg x \vee \neg y$.

The function nor_2 from *Boolean*^{2} into *Boolean* is defined by:

(Def. 10) For all elements x, y of *Boolean* holds $\mathrm{nor}_2(\langle x, y \rangle) = \neg(x \vee y)$.

The function nor_{2a} from *Boolean*^{2} into *Boolean* is defined by:

(Def. 11) For all elements x, y of *Boolean* holds $(\mathrm{nor}_{2a})(\langle x, y \rangle) = \neg(\neg x \vee y)$.

The function nor_{2b} from *Boolean*^{2} into *Boolean* is defined as follows:

(Def. 12) For all elements x, y of *Boolean* holds $(\mathrm{nor}_{2b})(\langle x, y \rangle) = \neg(\neg x \vee \neg y)$.

The function xor_2 from *Boolean*^{2} into *Boolean* is defined by:

(Def. 13) For all elements x, y of *Boolean* holds $\mathrm{xor}_2(\langle x, y \rangle) = x \oplus y$.

The function xor_{2a} from *Boolean*^{2} into *Boolean* is defined as follows:

(Def. 14) For all elements x, y of *Boolean* holds $(\mathrm{xor}_{2a})(\langle x, y \rangle) = \neg x \oplus y$.

The function xor_{2b} from *Boolean*^{2} into *Boolean* is defined as follows:

(Def. 15) For all elements x, y of *Boolean* holds $(\mathrm{xor}_{2b})(\langle x, y \rangle) = \neg x \oplus \neg y$.

We now state a number of propositions:

(3) For all elements x, y of *Boolean* holds $\mathrm{and}_2(\langle x, y \rangle) = x \wedge y$ and $(\mathrm{and}_{2a})(\langle x, y \rangle) = \neg x \wedge y$ and $(\mathrm{and}_{2b})(\langle x, y \rangle) = \neg x \wedge \neg y$.

(4) For all elements x, y of *Boolean* holds $\mathrm{nand}_2(\langle x, y \rangle) = \neg(x \wedge y)$ and $(\mathrm{nand}_{2a})(\langle x, y \rangle) = \neg(\neg x \wedge y)$ and $(\mathrm{nand}_{2b})(\langle x, y \rangle) = \neg(\neg x \wedge \neg y)$.

(5) For all elements x, y of *Boolean* holds $\mathrm{or}_2(\langle x, y \rangle) = x \vee y$ and $(\mathrm{or}_{2a})(\langle x, y \rangle) = \neg x \vee y$ and $(\mathrm{or}_{2b})(\langle x, y \rangle) = \neg x \vee \neg y$.

(6) For all elements x, y of *Boolean* holds $\mathrm{nor}_2(\langle x, y\rangle) = \neg(x \vee y)$ and $(\mathrm{nor}_{2a})(\langle x, y\rangle) = \neg(\neg x \vee y)$ and $(\mathrm{nor}_{2b})(\langle x, y\rangle) = \neg(\neg x \vee \neg y)$.

(7) For all elements x, y of *Boolean* holds $\mathrm{xor}_2(\langle x, y\rangle) = x \oplus y$ and $(\mathrm{xor}_{2a})(\langle x, y\rangle) = \neg x \oplus y$ and $(\mathrm{xor}_{2b})(\langle x, y\rangle) = \neg x \oplus \neg y$.

(8) For all elements x, y of *Boolean* holds $\mathrm{and}_2(\langle x, y\rangle) = (\mathrm{nor}_{2b})(\langle x, y\rangle)$ and $(\mathrm{and}_{2a})(\langle x, y\rangle) = (\mathrm{nor}_{2a})(\langle y, x\rangle)$ and $(\mathrm{and}_{2b})(\langle x, y\rangle) = \mathrm{nor}_2(\langle x, y\rangle)$.

(9) For all elements x, y of *Boolean* holds $\mathrm{or}_2(\langle x, y\rangle) = (\mathrm{nand}_{2b})(\langle x, y\rangle)$ and $(\mathrm{or}_{2a})(\langle x, y\rangle) = (\mathrm{nand}_{2a})(\langle y, x\rangle)$ and $(\mathrm{or}_{2b})(\langle x, y\rangle) = \mathrm{nand}_2(\langle x, y\rangle)$.

(10) For all elements x, y of *Boolean* holds $(\mathrm{xor}_{2b})(\langle x, y\rangle) = \mathrm{xor}_2(\langle x, y\rangle)$.

(11)(i) $\mathrm{and}_2(\langle 0, 0\rangle) = 0$,
(ii) $\mathrm{and}_2(\langle 0, 1\rangle) = 0$,
(iii) $\mathrm{and}_2(\langle 1, 0\rangle) = 0$,
(iv) $\mathrm{and}_2(\langle 1, 1\rangle) = 1$,
(v) $(\mathrm{and}_{2a})(\langle 0, 0\rangle) = 0$,
(vi) $(\mathrm{and}_{2a})(\langle 0, 1\rangle) = 1$,
(vii) $(\mathrm{and}_{2a})(\langle 1, 0\rangle) = 0$,
(viii) $(\mathrm{and}_{2a})(\langle 1, 1\rangle) = 0$,
(ix) $(\mathrm{and}_{2b})(\langle 0, 0\rangle) = 1$,
(x) $(\mathrm{and}_{2b})(\langle 0, 1\rangle) = 0$,
(xi) $(\mathrm{and}_{2b})(\langle 1, 0\rangle) = 0$, and
(xii) $(\mathrm{and}_{2b})(\langle 1, 1\rangle) = 0$.

(12)(i) $\mathrm{or}_2(\langle 0, 0\rangle) = 0$,
(ii) $\mathrm{or}_2(\langle 0, 1\rangle) = 1$,
(iii) $\mathrm{or}_2(\langle 1, 0\rangle) = 1$,
(iv) $\mathrm{or}_2(\langle 1, 1\rangle) = 1$,
(v) $(\mathrm{or}_{2a})(\langle 0, 0\rangle) = 1$,
(vi) $(\mathrm{or}_{2a})(\langle 0, 1\rangle) = 1$,
(vii) $(\mathrm{or}_{2a})(\langle 1, 0\rangle) = 0$,
(viii) $(\mathrm{or}_{2a})(\langle 1, 1\rangle) = 1$,
(ix) $(\mathrm{or}_{2b})(\langle 0, 0\rangle) = 1$,
(x) $(\mathrm{or}_{2b})(\langle 0, 1\rangle) = 1$,
(xi) $(\mathrm{or}_{2b})(\langle 1, 0\rangle) = 1$, and
(xii) $(\mathrm{or}_{2b})(\langle 1, 1\rangle) = 0$.

(13) $\mathrm{xor}_2(\langle 0, 0\rangle) = 0$ and $\mathrm{xor}_2(\langle 0, 1\rangle) = 1$ and $\mathrm{xor}_2(\langle 1, 0\rangle) = 1$ and $\mathrm{xor}_2(\langle 1, 1\rangle) = 0$ and $(\mathrm{xor}_{2a})(\langle 0, 0\rangle) = 1$ and $(\mathrm{xor}_{2a})(\langle 0, 1\rangle) = 0$ and $(\mathrm{xor}_{2a})(\langle 1, 0\rangle) = 0$ and $(\mathrm{xor}_{2a})(\langle 1, 1\rangle) = 1$.

The function and_3 from $\mathit{Boolean}^3$ into *Boolean* is defined as follows:

(Def. 16) For all elements x, y, z of *Boolean* holds $\mathrm{and}_3(\langle x, y, z\rangle) = x \wedge y \wedge z$.

The function and_{3a} from $\mathit{Boolean}^3$ into *Boolean* is defined by:

(Def. 17) For all elements x, y, z of *Boolean* holds $(\mathrm{and}_{3a})(\langle x, y, z\rangle) = \neg x \wedge y \wedge z$.

The function and_{3b} from $\mathit{Boolean}^3$ into *Boolean* is defined by:

(Def. 18) For all elements x, y, z of *Boolean* holds $(\mathrm{and}_{3b})(\langle x, y, z\rangle) = \neg x \wedge \neg y \wedge z$.

The function and_{3c} from $\mathit{Boolean}^3$ into *Boolean* is defined by:

(Def. 19) For all elements x, y, z of *Boolean* holds $(\mathrm{and}_{3c})(\langle x, y, z\rangle) = \neg x \wedge \neg y \wedge \neg z$.

The function nand_3 from *Boolean*^{3} into *Boolean* is defined by:

(Def. 20) For all elements x, y, z of *Boolean* holds $\mathrm{nand}_3(\langle x, y, z\rangle) = \neg(x \wedge y \wedge z)$.

The function nand_{3a} from *Boolean*^{3} into *Boolean* is defined as follows:

(Def. 21) For all elements x, y, z of *Boolean* holds $(\mathrm{nand}_{3a})(\langle x, y, z\rangle) = \neg(\neg x \wedge y \wedge z)$.

The function nand_{3b} from *Boolean*^{3} into *Boolean* is defined as follows:

(Def. 22) For all elements x, y, z of *Boolean* holds $(\mathrm{nand}_{3b})(\langle x, y, z\rangle) = \neg(\neg x \wedge \neg y \wedge z)$.

The function nand_{3c} from *Boolean*^{3} into *Boolean* is defined by:

(Def. 23) For all elements x, y, z of *Boolean* holds $(\mathrm{nand}_{3c})(\langle x, y, z\rangle) = \neg(\neg x \wedge \neg y \wedge \neg z)$.

The function or_3 from *Boolean*^{3} into *Boolean* is defined by:

(Def. 24) For all elements x, y, z of *Boolean* holds $\mathrm{or}_3(\langle x, y, z\rangle) = x \vee y \vee z$.

The function or_{3a} from *Boolean*^{3} into *Boolean* is defined as follows:

(Def. 25) For all elements x, y, z of *Boolean* holds $(\mathrm{or}_{3a})(\langle x, y, z\rangle) = \neg x \vee y \vee z$.

The function or_{3b} from *Boolean*^{3} into *Boolean* is defined as follows:

(Def. 26) For all elements x, y, z of *Boolean* holds $(\mathrm{or}_{3b})(\langle x, y, z\rangle) = \neg x \vee \neg y \vee z$.

The function or_{3c} from *Boolean*^{3} into *Boolean* is defined as follows:

(Def. 27) For all elements x, y, z of *Boolean* holds $(\mathrm{or}_{3c})(\langle x, y, z\rangle) = \neg x \vee \neg y \vee \neg z$.

The function nor_3 from *Boolean*^{3} into *Boolean* is defined by:

(Def. 28) For all elements x, y, z of *Boolean* holds $\mathrm{nor}_3(\langle x, y, z\rangle) = \neg(x \vee y \vee z)$.

The function nor_{3a} from *Boolean*^{3} into *Boolean* is defined as follows:

(Def. 29) For all elements x, y, z of *Boolean* holds $(\mathrm{nor}_{3a})(\langle x, y, z\rangle) = \neg(\neg x \vee y \vee z)$.

The function nor_{3b} from *Boolean*^{3} into *Boolean* is defined by:

(Def. 30) For all elements x, y, z of *Boolean* holds $(\mathrm{nor}_{3b})(\langle x, y, z\rangle) = \neg(\neg x \vee \neg y \vee z)$.

The function nor_{3c} from *Boolean*^{3} into *Boolean* is defined by:

(Def. 31) For all elements x, y, z of *Boolean* holds $(\mathrm{nor}_{3c})(\langle x, y, z\rangle) = \neg(\neg x \vee \neg y \vee \neg z)$.

The function xor_3 from *Boolean*^{3} into *Boolean* is defined by:

(Def. 32) For all elements x, y, z of *Boolean* holds $\mathrm{xor}_3(\langle x, y, z\rangle) = x \oplus y \oplus z$.

Next we state a number of propositions:

(14) For all elements x, y, z of *Boolean* holds $\mathrm{and}_3(\langle x, y, z\rangle) = x \wedge y \wedge z$ and $(\mathrm{and}_{3a})(\langle x, y, z\rangle) = \neg x \wedge y \wedge z$ and $(\mathrm{and}_{3b})(\langle x, y, z\rangle) = \neg x \wedge \neg y \wedge z$ and $(\mathrm{and}_{3c})(\langle x, y, z\rangle) = \neg x \wedge \neg y \wedge \neg z$.

(15) Let x, y, z be elements of *Boolean*. Then $\mathrm{nand}_3(\langle x, y, z\rangle) = \neg(x \wedge y \wedge z)$ and $(\mathrm{nand}_{3a})(\langle x, y, z\rangle) = \neg(\neg x \wedge y \wedge z)$ and $(\mathrm{nand}_{3b})(\langle x, y, z\rangle) = \neg(\neg x \wedge \neg y \wedge z)$ and $(\mathrm{nand}_{3c})(\langle x, y, z\rangle) = \neg(\neg x \wedge \neg y \wedge \neg z)$.

(16) For all elements x, y, z of *Boolean* holds $\mathrm{or}_3(\langle x, y, z\rangle) = x \vee y \vee z$ and $(\mathrm{or}_{3a})(\langle x, y, z\rangle) = \neg x \vee y \vee z$ and $(\mathrm{or}_{3b})(\langle x, y, z\rangle) = \neg x \vee \neg y \vee z$ and $(\mathrm{or}_{3c})(\langle x, y, z\rangle) = \neg x \vee \neg y \vee \neg z$.

(17) Let x, y, z be elements of *Boolean*. Then $\mathrm{nor}_3(\langle x, y, z\rangle) = \neg(x \vee y \vee z)$ and $(\mathrm{nor}_{3a})(\langle x, y, z\rangle) = \neg(\neg x \vee y \vee z)$ and $(\mathrm{nor}_{3b})(\langle x, y, z\rangle) = \neg(\neg x \vee \neg y \vee z)$ and $(\mathrm{nor}_{3c})(\langle x, y, z\rangle) = \neg(\neg x \vee \neg y \vee \neg z)$.

(18) For all elements x, y, z of *Boolean* holds $\mathrm{xor}_3(\langle x, y, z\rangle) = x \oplus y \oplus z$.

(19) For all elements x, y, z of *Boolean* holds $\mathrm{and}_3(\langle x, y, z\rangle) = (\mathrm{nor}_{3c})(\langle x, y, z\rangle)$ and $(\mathrm{and}_{3a})(\langle x, y, z\rangle) = (\mathrm{nor}_{3b})(\langle z, y, x\rangle)$ and $(\mathrm{and}_{3b})(\langle x, y, z\rangle) = (\mathrm{nor}_{3a})(\langle z, y, x\rangle)$ and $(\mathrm{and}_{3c})(\langle x, y, z\rangle) = \mathrm{nor}_3(\langle x, y, z\rangle)$.

(20) For all elements x, y, z of *Boolean* holds $\mathrm{or}_3(\langle x, y, z\rangle) = (\mathrm{nand}_{3c})(\langle x, y, z\rangle)$ and $(\mathrm{or}_{3a})(\langle x, y, z\rangle) = (\mathrm{nand}_{3b})(\langle z, y, x\rangle)$ and $(\mathrm{or}_{3b})(\langle x, y, z\rangle) = (\mathrm{nand}_{3a})(\langle z, y, x\rangle)$ and $(\mathrm{or}_{3c})(\langle x, y, z\rangle) = \mathrm{nand}_3(\langle x, y, z\rangle)$.

(21) $\mathrm{and}_3(\langle 0, 0, 0\rangle) = 0$ and $\mathrm{and}_3(\langle 0, 0, 1\rangle) = 0$ and $\mathrm{and}_3(\langle 0, 1, 0\rangle) = 0$ and $\mathrm{and}_3(\langle 0, 1, 1\rangle) = 0$ and $\mathrm{and}_3(\langle 1, 0, 0\rangle) = 0$ and $\mathrm{and}_3(\langle 1, 0, 1\rangle) = 0$ and $\mathrm{and}_3(\langle 1, 1, 0\rangle) = 0$ and $\mathrm{and}_3(\langle 1, 1, 1\rangle) = 1$.

(22) $(\mathrm{and}_{3a})(\langle 0, 0, 0\rangle) = 0$ and $(\mathrm{and}_{3a})(\langle 0, 0, 1\rangle) = 0$ and $(\mathrm{and}_{3a})(\langle 0, 1, 0\rangle) = 0$ and $(\mathrm{and}_{3a})(\langle 0, 1, 1\rangle) = 1$ and $(\mathrm{and}_{3a})(\langle 1, 0, 0\rangle) = 0$ and $(\mathrm{and}_{3a})(\langle 1, 0, 1\rangle) = 0$ and $(\mathrm{and}_{3a})(\langle 1, 1, 0\rangle) = 0$ and $(\mathrm{and}_{3a})(\langle 1, 1, 1\rangle) = 0$.

(23) $(\mathrm{and}_{3b})(\langle 0, 0, 0\rangle) = 0$ and $(\mathrm{and}_{3b})(\langle 0, 0, 1\rangle) = 1$ and $(\mathrm{and}_{3b})(\langle 0, 1, 0\rangle) = 0$ and $(\mathrm{and}_{3b})(\langle 0, 1, 1\rangle) = 0$ and $(\mathrm{and}_{3b})(\langle 1, 0, 0\rangle) = 0$ and $(\mathrm{and}_{3b})(\langle 1, 0, 1\rangle) = 0$ and $(\mathrm{and}_{3b})(\langle 1, 1, 0\rangle) = 0$ and $(\mathrm{and}_{3b})(\langle 1, 1, 1\rangle) = 0$.

(24) $(\mathrm{and}_{3c})(\langle 0, 0, 0\rangle) = 1$ and $(\mathrm{and}_{3c})(\langle 0, 0, 1\rangle) = 0$ and $(\mathrm{and}_{3c})(\langle 0, 1, 0\rangle) = 0$ and $(\mathrm{and}_{3c})(\langle 0, 1, 1\rangle) = 0$ and $(\mathrm{and}_{3c})(\langle 1, 0, 0\rangle) = 0$ and $(\mathrm{and}_{3c})(\langle 1, 0, 1\rangle) = 0$ and $(\mathrm{and}_{3c})(\langle 1, 1, 0\rangle) = 0$ and $(\mathrm{and}_{3c})(\langle 1, 1, 1\rangle) = 0$.

(25) $\mathrm{or}_3(\langle 0, 0, 0\rangle) = 0$ and $\mathrm{or}_3(\langle 0, 0, 1\rangle) = 1$ and $\mathrm{or}_3(\langle 0, 1, 0\rangle) = 1$ and $\mathrm{or}_3(\langle 0, 1, 1\rangle) = 1$ and $\mathrm{or}_3(\langle 1, 0, 0\rangle) = 1$ and $\mathrm{or}_3(\langle 1, 0, 1\rangle) = 1$ and $\mathrm{or}_3(\langle 1, 1, 0\rangle) = 1$ and $\mathrm{or}_3(\langle 1, 1, 1\rangle) = 1$.

(26) $(\mathrm{or}_{3a})(\langle 0, 0, 0\rangle) = 1$ and $(\mathrm{or}_{3a})(\langle 0, 0, 1\rangle) = 1$ and $(\mathrm{or}_{3a})(\langle 0, 1, 0\rangle) = 1$ and $(\mathrm{or}_{3a})(\langle 0, 1, 1\rangle) = 1$ and $(\mathrm{or}_{3a})(\langle 1, 0, 0\rangle) = 0$ and $(\mathrm{or}_{3a})(\langle 1, 0, 1\rangle) = 1$ and $(\mathrm{or}_{3a})(\langle 1, 1, 0\rangle) = 1$ and $(\mathrm{or}_{3a})(\langle 1, 1, 1\rangle) = 1$.

(27) $(\mathrm{or}_{3b})(\langle 0, 0, 0\rangle) = 1$ and $(\mathrm{or}_{3b})(\langle 0, 0, 1\rangle) = 1$ and $(\mathrm{or}_{3b})(\langle 0, 1, 0\rangle) = 1$ and $(\mathrm{or}_{3b})(\langle 0, 1, 1\rangle) = 1$ and $(\mathrm{or}_{3b})(\langle 1, 0, 0\rangle) = 1$ and $(\mathrm{or}_{3b})(\langle 1, 0, 1\rangle) = 1$ and $(\mathrm{or}_{3b})(\langle 1, 1, 0\rangle) = 0$ and $(\mathrm{or}_{3b})(\langle 1, 1, 1\rangle) = 1$.

(28) $(\mathrm{or}_{3c})(\langle 0, 0, 0\rangle) = 1$ and $(\mathrm{or}_{3c})(\langle 0, 0, 1\rangle) = 1$ and $(\mathrm{or}_{3c})(\langle 0, 1, 0\rangle) = 1$ and $(\mathrm{or}_{3c})(\langle 0, 1, 1\rangle) = 1$ and $(\mathrm{or}_{3c})(\langle 1, 0, 0\rangle) = 1$ and $(\mathrm{or}_{3c})(\langle 1, 0, 1\rangle) = 1$ and $(\mathrm{or}_{3c})(\langle 1, 1, 0\rangle) = 1$ and $(\mathrm{or}_{3c})(\langle 1, 1, 1\rangle) = 0$.

(29) $\mathrm{xor}_3(\langle 0, 0, 0\rangle) = 0$ and $\mathrm{xor}_3(\langle 0, 0, 1\rangle) = 1$ and $\mathrm{xor}_3(\langle 0, 1, 0\rangle) = 1$ and $\mathrm{xor}_3(\langle 0, 1, 1\rangle) = 0$ and $\mathrm{xor}_3(\langle 1, 0, 0\rangle) = 1$ and $\mathrm{xor}_3(\langle 1, 0, 1\rangle) = 0$ and $\mathrm{xor}_3(\langle 1, 1, 0\rangle) = 0$ and $\mathrm{xor}_3(\langle 1, 1, 1\rangle) = 1$.

2. 2's Complement Circuit Properties

Let x, b be sets. The functor $\mathrm{CompStr}(x, b)$ yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined by:

(Def. 33) $\mathrm{CompStr}(x, b) = \mathrm{1GateCircStr}(\langle x, b\rangle, \mathrm{xor}_{2a})$.

Let x, b be sets. The functor $\mathrm{CompCirc}(x, b)$ yields a strict Boolean circuit of $\mathrm{CompStr}(x, b)$ with denotation held in gates and is defined as follows:

(Def. 34) $\mathrm{CompCirc}(x, b) = \mathrm{1GateCircuit}(x, b, \mathrm{xor}_{2a})$.

Let x, b be sets. The functor $\mathrm{CompOutput}(x, b)$ yielding an element of $\mathrm{InnerVertices}(\mathrm{CompStr}(x, b))$ is defined by:

(Def. 35) $\mathrm{CompOutput}(x, b) = \langle\!\langle \langle x, b\rangle, \mathrm{xor}_{2a} \rangle\!\rangle$.

Let x, b be sets. The functor $\mathrm{IncrementStr}(x, b)$ yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 36) $\mathrm{IncrementStr}(x, b) = \mathrm{1GateCircStr}(\langle x, b\rangle, \mathrm{and}_{2a})$.

Let x, b be sets. The functor $\mathrm{IncrementCirc}(x, b)$ yields a strict Boolean circuit of $\mathrm{IncrementStr}(x, b)$ with denotation held in gates and is defined as follows:

(Def. 37) $\mathrm{IncrementCirc}(x, b) = \mathrm{1GateCircuit}(x, b, \mathrm{and}_{2a})$.

Let x, b be sets. The functor $\mathrm{IncrementOutput}(x, b)$ yields an element of $\mathrm{InnerVertices}(\mathrm{IncrementStr}(x, b))$ and is defined by:

(Def. 38) $\mathrm{IncrementOutput}(x, b) = \langle\!\langle \langle x, b\rangle, \mathrm{and}_{2a} \rangle\!\rangle$.

Let x, b be sets. The functor $\mathrm{BitCompStr}(x, b)$ yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined as follows:

(Def. 39) $\mathrm{BitCompStr}(x, b) = \mathrm{CompStr}(x, b) {+}\!\cdot \mathrm{IncrementStr}(x, b)$.

Let x, b be sets. The functor $\mathrm{BitCompCirc}(x, b)$ yielding a strict Boolean circuit of $\mathrm{BitCompStr}(x, b)$ with denotation held in gates is defined by:

(Def. 40) $\mathrm{BitCompCirc}(x, b) = \mathrm{CompCirc}(x, b) {+}\!\cdot \mathrm{IncrementCirc}(x, b)$.

One can prove the following propositions:

(30) For all non pair sets x, b holds $\mathrm{InnerVertices}(\mathrm{CompStr}(x, b))$ is a binary relation.

(31) For all non pair sets x, b holds $x \in$ the carrier of $\mathrm{CompStr}(x, b)$ and $b \in$ the carrier of $\mathrm{CompStr}(x, b)$ and $\langle\!\langle \langle x, b\rangle, \mathrm{xor}_{2a} \rangle\!\rangle \in$ the carrier of $\mathrm{CompStr}(x, b)$.

(32) For all non pair sets x, b holds the carrier of $\mathrm{CompStr}(x, b) = \{x, b\} \cup \{\langle\!\langle \langle x, b\rangle, \mathrm{xor}_{2a} \rangle\!\rangle\}$.

(33) For all non pair sets x, b holds $\mathrm{InnerVertices}(\mathrm{CompStr}(x, b)) = \{\langle\!\langle \langle x, b\rangle, \mathrm{xor}_{2a} \rangle\!\rangle\}$.

(34) For all non pair sets x, b holds $\langle\!\langle \langle x, b\rangle, \mathrm{xor}_{2a} \rangle\!\rangle \in$ InnerVertices(CompStr(x, b)).

(35) For all non pair sets x, b holds InputVertices(CompStr(x, b)) $= \{x, b\}$.

(36) For all non pair sets x, b holds $x \in$ InputVertices(CompStr(x, b)) and $b \in$ InputVertices(CompStr(x, b)).

(37) For all non pair sets x, b holds InputVertices(CompStr(x, b)) has no pairs.

(38) For all non pair sets x, b holds InnerVertices(IncrementStr(x, b)) is a binary relation.

(39) For all non pair sets x, b holds $x \in$ the carrier of IncrementStr(x, b) and $b \in$ the carrier of IncrementStr(x, b) and $\langle\!\langle \langle x, b\rangle, \mathrm{and}_{2a} \rangle\!\rangle \in$ the carrier of IncrementStr(x, b).

(40) For all non pair sets x, b holds the carrier of IncrementStr$(x, b) = \{x, b\} \cup \{\langle\!\langle \langle x, b\rangle, \mathrm{and}_{2a} \rangle\!\rangle\}$.

(41) For all non pair sets x, b holds InnerVertices(IncrementStr(x, b)) $= \{\langle\!\langle \langle x, b\rangle, \mathrm{and}_{2a} \rangle\!\rangle\}$.

(42) For all non pair sets x, b holds $\langle\!\langle \langle x, b\rangle, \mathrm{and}_{2a} \rangle\!\rangle \in$ InnerVertices(IncrementStr(x, b)).

(43) For all non pair sets x, b holds InputVertices(IncrementStr(x, b)) $= \{x, b\}$.

(44) For all non pair sets x, b holds $x \in$ InputVertices(IncrementStr(x, b)) and $b \in$ InputVertices(IncrementStr(x, b)).

(45) For all non pair sets x, b holds InputVertices(IncrementStr(x, b)) has no pairs.

(46) For all non pair sets x, b holds InnerVertices(BitCompStr(x, b)) is a binary relation.

(47) Let x, b be non pair sets. Then
  (i) $x \in$ the carrier of BitCompStr(x, b),
  (ii) $b \in$ the carrier of BitCompStr(x, b),
  (iii) $\langle\!\langle \langle x, b\rangle, \mathrm{xor}_{2a} \rangle\!\rangle \in$ the carrier of BitCompStr(x, b), and
  (iv) $\langle\!\langle \langle x, b\rangle, \mathrm{and}_{2a} \rangle\!\rangle \in$ the carrier of BitCompStr(x, b).

(48) For all non pair sets x, b holds the carrier of BitCompStr$(x, b) = \{x, b\} \cup \{\langle\!\langle \langle x, b\rangle, \mathrm{xor}_{2a} \rangle\!\rangle, \langle\!\langle \langle x, b\rangle, \mathrm{and}_{2a} \rangle\!\rangle\}$.

(49) For all non pair sets x, b holds InnerVertices(BitCompStr(x, b)) $= \{\langle\!\langle \langle x, b\rangle, \mathrm{xor}_{2a} \rangle\!\rangle, \langle\!\langle \langle x, b\rangle, \mathrm{and}_{2a} \rangle\!\rangle\}$.

(50) For all non pair sets x, b holds $\langle\!\langle \langle x, b\rangle, \mathrm{xor}_{2a} \rangle\!\rangle \in$ InnerVertices(BitCompStr(x, b)) and $\langle\!\langle \langle x, b\rangle, \mathrm{and}_{2a} \rangle\!\rangle \in$ InnerVertices(BitCompStr(x, b)).

(51) For all non pair sets x, b holds InputVertices(BitCompStr(x, b)) $= \{x, b\}$.

(52) For all non pair sets x, b holds $x \in$ InputVertices(BitCompStr(x, b)) and $b \in$ InputVertices(BitCompStr(x, b)).

(53) For all non pair sets x, b holds InputVertices(BitCompStr(x, b)) has no pairs.

(54) For all non pair sets x, b and for every state s of CompCirc(x,b) holds (Following(s))(CompOutput(x,b)) $=$ $(\mathrm{xor}_{2a})(\langle s(x), s(b)\rangle)$ and (Following(s))$(x) = s(x)$ and (Following(s))$(b) = s(b)$.

(55) Let x, b be non pair sets, s be a state of CompCirc(x,b), and a_1, a_2 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(b)$, then (Following(s))(CompOutput(x,b)) $= \neg a_1 \oplus a_2$ and (Following(s))$(x) = a_1$ and (Following(s))$(b) = a_2$.

(56) For all non pair sets x, b and for every state s of BitCompCirc(x,b) holds (Following(s))(CompOutput(x,b)) $=$ $(\mathrm{xor}_{2a})(\langle s(x), s(b)\rangle)$ and (Following(s))$(x) = s(x)$ and (Following(s))$(b) = s(b)$.

(57) Let x, b be non pair sets, s be a state of BitCompCirc(x,b), and a_1, a_2 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(b)$, then (Following(s))(CompOutput(x,b)) $= \neg a_1 \oplus a_2$ and (Following(s))$(x) = a_1$ and (Following(s))$(b) = a_2$.

(58) For all non pair sets x, b and for every state s of IncrementCirc(x,b) holds (Following(s))(IncrementOutput(x,b)) $=$ $(\mathrm{and}_{2a})(\langle s(x), s(b)\rangle)$ and (Following(s))$(x) = s(x)$ and (Following(s))$(b) = s(b)$.

(59) Let x, b be non pair sets, s be a state of IncrementCirc(x,b), and a_1, a_2 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(b)$, then (Following(s))(IncrementOutput(x,b)) $= \neg a_1 \wedge a_2$ and (Following(s))$(x) = a_1$ and (Following(s))$(b) = a_2$.

(60) For all non pair sets x, b and for every state s of BitCompCirc(x,b) holds (Following(s))(IncrementOutput(x,b)) $=$ $(\mathrm{and}_{2a})(\langle s(x), s(b)\rangle)$ and (Following(s))$(x) = s(x)$ and (Following(s))$(b) = s(b)$.

(61) Let x, b be non pair sets, s be a state of BitCompCirc(x,b), and a_1, a_2 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(b)$, then (Following(s))(IncrementOutput(x,b)) $= \neg a_1 \wedge a_2$ and (Following(s))$(x) = a_1$ and (Following(s))$(b) = a_2$.

(62) Let x, b be non pair sets and s be a state of BitCompCirc(x,b). Then (Following(s))(CompOutput(x,b)) $=$ $(\mathrm{xor}_{2a})(\langle s(x), s(b)\rangle)$ and (Following(s))(IncrementOutput(x,b)) $= (\mathrm{and}_{2a})(\langle s(x), s(b)\rangle)$ and (Following(s))$(x) = s(x)$ and (Following(s))$(b) = s(b)$.

(63) Let x, b be non pair sets, s be a state of BitCompCirc(x,b), and a_1, a_2 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(b)$. Then (Following(s))(CompOutput(x,b)) $= \neg a_1 \oplus a_2$ and (Following(s))(IncrementOutput(x,b)) $= \neg a_1 \wedge a_2$ and (Following(s))$(x) = a_1$ and (Following(s))$(b) = a_2$.

(64) For all non pair sets x, b and for every state s of BitCompCirc(x,b) holds Following(s) is stable.

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The Equational Characterization of Continuous Lattices[1]

Mariusz Żynel
Warsaw University
Białystok

Summary. The class of continuous lattices can be characterized by infinitary equations. Therefore, it is closed under the formation of subalgebras and homomorphic images. Following the terminology of [18] we introduce a continuous lattice subframe to be a sublattice closed under the formation of arbitrary infs and directed sups. This notion corresponds with a subalgebra of a continuous lattice in [16].

The class of completely distributive lattices is also introduced in the paper. Such lattices are complete and satisfy the most restrictive type of the general distributivity law. Obviously each completely distributive lattice is a Heyting algebra. It was hard to find the best Mizar implementation of the complete distributivity equational condition (denoted by CD in [16]). The powerful and well developed Many Sorted Theory gives the most convenient way of this formalization. A set double indexed by K, introduced in the paper, corresponds with a family $\{x_{j,k} : j \in J, k \in K(j)\}$. It is defined to be a suitable many sorted function. Two special functors: Sups and Infs as counterparts of Sup and Inf respectively, introduced in [38], are also defined. Originally the equation in Definition 2.4 of [16, p. 58] looks as follows:

$$\bigwedge_{j\in J}\bigvee_{k\in K(j)}x_{j,k} = \bigvee_{f\in M}\bigwedge_{j\in J}x_{j,f(j)},$$

where M is the set of functions defined on J with values $f(j) \in K(j)$.

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The articles [30], [37], [12], [15], [35], [10], [11], [1], [4], [29], [36], [5], [2], [28], [13], [9], [32], [21], [22], [33], [19], [24], [27], [20], [25], [31], [3], [26], [23], [6], [17], [38], [14], [7], [8], and [34] provide the terminology and notation for this paper.

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1. The Continuity of Lattices

In this paper x, y are arbitrary, X denotes a set, and L denotes an up-complete semilattice.

One can prove the following propositions:

(1) L is continuous if and only if for every element x of L holds $\downarrow x$ is an ideal of L and $x \leqslant \sup \downarrow x$ and for every ideal I of L such that $x \leqslant \sup I$ holds $\downarrow x \subseteq I$.

(2) L is continuous if and only if for every element x of L there exists an ideal I of L such that $x \leqslant \sup I$ and for every ideal J of L such that $x \leqslant \sup J$ holds $I \subseteq J$.

(3) For every continuous lower-bounded sup-semilattice L holds $\mathrm{SupMap}(L)$ has a lower adjoint.

(4) For every up-complete lower-bounded lattice L such that $\mathrm{SupMap}(L)$ is upper adjoint holds L is continuous.

(5) For every complete semilattice L such that $\mathrm{SupMap}(L)$ is infs-preserving and sups-preserving holds $\mathrm{SupMap}(L)$ has a lower adjoint.

Let J, D be sets and let K be a many sorted set indexed by J. A set of elements of D double indexed by K is a many sorted function from K into $J \longmapsto D$.

Let J be a set, let K be a many sorted set indexed by J, and let S be a 1-sorted structure. A set of elements of S double indexed by K is a set of elements of the carrier of S double indexed by K.

We now state the proposition

(6) Let J, D be sets, K be a many sorted set indexed by J, F be a set of elements of D double indexed by K, and j be arbitrary. If $j \in J$, then $F(j)$ is a function from $K(j)$ into D.

Let J, D be non empty sets, let K be a many sorted set indexed by J, let F be a set of elements of D double indexed by K, and let j be an element of J. Then $F(j)$ is a function from $K(j)$ into D.

Let J, D be non empty sets, let K be a non-empty many sorted set indexed by J, let F be a set of elements of D double indexed by K, and let j be an element of J. One can check that $\operatorname{rng} F(j)$ is non empty.

Let J be a set, let D be a non empty set, and let K be a non-empty many sorted set indexed by J. One can check that every set of elements of D double indexed by K is non-empty.

Next we state four propositions:

(7) For every function yielding function F and for arbitrary f such that $f \in \operatorname{dom} \mathrm{Frege}(F)$ holds f is a function.

(8) For every function yielding function F and for every function f such that $f \in \operatorname{dom} \mathrm{Frege}(F)$ holds $\operatorname{dom} f = \operatorname{dom} F$ and $\operatorname{dom} F = \operatorname{dom}(\mathrm{Frege}(F))(f)$.

(9) Let F be a function yielding function and f be a function. Suppose $f \in \operatorname{dom} \operatorname{Frege}(F)$. Let i be arbitrary. If $i \in \operatorname{dom} F$, then $f(i) \in \operatorname{dom} F(i)$ and $(\operatorname{Frege}(F))(f)(i) = F(i)(f(i))$ and $F(i)(f(i)) \in \operatorname{rng}(\operatorname{Frege}(F))(f)$.

(10) Let J, D be sets, K be a many sorted set indexed by J, F be a set of elements of D double indexed by K, and f be a function. If $f \in \operatorname{dom} \operatorname{Frege}(F)$, then $(\operatorname{Frege}(F))(f)$ is a function from J into D.

Let f be a non-empty function. Note that $\operatorname{dom}_\kappa f(\kappa)$ is non-empty.

Let J, D be sets, let K be a many sorted set indexed by J, and let F be a set of elements of D double indexed by K. Then $\operatorname{Frege}(F)$ is a set of elements of D double indexed by $\prod(\operatorname{dom}_\kappa F(\kappa)) \longmapsto J$.

Let J, D be non empty sets, let K be a non-empty many sorted set indexed by J, let F be a set of elements of D double indexed by K, let G be a set of elements of D double indexed by $\prod(\operatorname{dom}_\kappa F(\kappa)) \longmapsto J$, and let f be an element of $\prod(\operatorname{dom}_\kappa F(\kappa))$. Then $G(f)$ is a function from J into D.

Let L be a non empty relational structure and let F be a function yielding function. The functor $\underline{\bigsqcup}_L F$ yields a function from $\operatorname{dom} F$ into the carrier of L and is defined as follows:

(Def. 1) For every x such that $x \in \operatorname{dom} F$ holds $(\underline{\bigsqcup}_L F)(x) = \bigsqcup_L F(x)$.

The functor $\overline{\bigsqcap}_L F$ yields a function from $\operatorname{dom} F$ into the carrier of L and is defined by:

(Def. 2) For every x such that $x \in \operatorname{dom} F$ holds $(\overline{\bigsqcap}_L F)(x) = \bigsqcap_L F(x)$.

Let J be a set, let K be a many sorted set indexed by J, let L be a non empty relational structure, and let F be a set of elements of L double indexed by K. We introduce $\operatorname{Sups}(F)$ as a synonym of $\underline{\bigsqcup}_L F$. We introduce $\operatorname{Infs}(F)$ as a synonym of $\overline{\bigsqcap}_L F$.

Let I, J be sets, let L be a non empty relational structure, and let F be a set of elements of L double indexed by $I \longmapsto J$. We introduce $\operatorname{Sups}(F)$ as a synonym of $\underline{\bigsqcup}_L F$. We introduce $\operatorname{Infs}(F)$ as a synonym of $\overline{\bigsqcap}_L F$.

The following four propositions are true:

(11) Let L be a non empty relational structure and F, G be function yielding functions. If $\operatorname{dom} F = \operatorname{dom} G$ and for every x such that $x \in \operatorname{dom} F$ holds $\bigsqcup_L F(x) = \bigsqcup_L G(x)$, then $\underline{\bigsqcup}_L F = \underline{\bigsqcup}_L G$.

(12) Let L be a non empty relational structure and F, G be function yielding functions. If $\operatorname{dom} F = \operatorname{dom} G$ and for every x such that $x \in \operatorname{dom} F$ holds $\bigsqcap_L F(x) = \bigsqcap_L G(x)$, then $\overline{\bigsqcap}_L F = \overline{\bigsqcap}_L G$.

(13) Let L be a non empty relational structure and F be a function yielding function. Then

(i) $y \in \operatorname{rng} \underline{\bigsqcup}_L F$ iff there exists x such that $x \in \operatorname{dom} F$ and $y = \bigsqcup_L F(x)$, and

(ii) $y \in \operatorname{rng} \overline{\bigsqcap}_L F$ iff there exists x such that $x \in \operatorname{dom} F$ and $y = \bigsqcap_L F(x)$.

(14) Let L be a non empty relational structure, J be a non empty set, K be a many sorted set indexed by J, and F be a set of elements of L double indexed by K. Then

(i) $x \in \operatorname{rng}\operatorname{Sups}(F)$ iff there exists an element j of J such that $x = \operatorname{Sup}(F(j))$, and

(ii) $x \in \operatorname{rng}\operatorname{Infs}(F)$ iff there exists an element j of J such that $x = \operatorname{Inf}(F(j))$.

Let J be a non empty set, let K be a many sorted set indexed by J, let L be a non empty relational structure, and let F be a set of elements of L double indexed by K. Observe that $\operatorname{rng}\operatorname{Sups}(F)$ is non empty and $\operatorname{rng}\operatorname{Infs}(F)$ is non empty.

For simplicity we follow the rules: L is a complete lattice, a, b, c are elements of L, J is a non empty set, and K is a non-empty many sorted set indexed by J.

One can prove the following propositions:

(15) Let F be a function yielding function. If for every function f such that $f \in \operatorname{dom}\operatorname{Frege}(F)$ holds $\bigsqcap_L(\operatorname{Frege}(F))(f) \leqslant a$, then $\operatorname{Sup}(\overline{\bigsqcap}_L \operatorname{Frege}(F)) \leqslant a$.

(16) For every set F of elements of L double indexed by K holds $\operatorname{Inf}(\operatorname{Sups}(F)) \geqslant \operatorname{Sup}(\operatorname{Infs}(\operatorname{Frege}(F)))$.

(17) If L is continuous and for every c such that $c \ll a$ holds $c \leqslant b$, then $a \leqslant b$.

(18) Suppose that for every non empty set J such that $J \in$ the universe of the carrier of L and for every non-empty many sorted set K indexed by J such that for every element j of J holds $K(j) \in$ the universe of the carrier of L and for every set F of elements of L double indexed by K such that for every element j of J holds $\operatorname{rng} F(j)$ is directed holds $\operatorname{Inf}(\operatorname{Sups}(F)) = \operatorname{Sup}(\operatorname{Infs}(\operatorname{Frege}(F)))$. Then L is continuous.

(19) L is continuous if and only if for all J, K and for every set F of elements of L double indexed by K such that for every element j of J holds $\operatorname{rng} F(j)$ is directed holds $\operatorname{Inf}(\operatorname{Sups}(F)) = \operatorname{Sup}(\operatorname{Infs}(\operatorname{Frege}(F)))$.

Let J, K, D be non empty sets and let F be a function from $[\!: J, K :\!]$ into D. Then $\operatorname{curry} F$ is a set of elements of D double indexed by $J \longmapsto K$.

We follow a convention: J, K, D will denote non empty sets, j will denote an element of J, and k will denote an element of K.

One can prove the following four propositions:

(20) For every function F from $[\!: J, K :\!]$ into D holds $\operatorname{dom}\operatorname{curry} F = J$ and $\operatorname{dom}(\operatorname{curry} F)(j) = K$ and $F(\langle j, k\rangle) = (\operatorname{curry} F)(j)(k)$.

(21) L is continuous if and only if for all non empty sets J, K and for every function F from $[\!: J, K :\!]$ into the carrier of L such that for every element j of J holds $\operatorname{rng}(\operatorname{curry} F)(j)$ is directed holds $\operatorname{Inf}(\operatorname{Sups}(\operatorname{curry} F)) = \operatorname{Sup}(\operatorname{Infs}(\operatorname{Frege}(\operatorname{curry} F)))$.

(22) Let F be a function from $[\!: J, K :\!]$ into the carrier of L and X be a subset of L. Suppose $X = \{a, a$ ranges over elements of L: $\bigvee_{f:\ \text{non-empty many sorted set indexed by } J}\ (f \in (\operatorname{Fin} K)^J \wedge \bigvee_{G:\text{set of elements of } L \text{ double indexed by } f}\ (\bigwedge_{j,x}\ (x \in f(j) \Rightarrow G(j)(x) =$

$F(\langle j, x\rangle)) \wedge a = \mathrm{Inf}(\mathrm{Sups}(G))))\}$. Then $\mathrm{Inf}(\mathrm{Sups}(\mathrm{curry}\, F)) \geqslant \sup X$.

(23) L is continuous if and only if for all J, K and for every function F from $[: J, K :]$ into the carrier of L and for every subset X of L such that $X = \{a, a$ ranges over elements of L: $\bigvee_{f:\text{ non-empty many sorted set indexed by } J} (f \in (\mathrm{Fin}\, K)^J \wedge \bigvee_{G:\text{set of elements of } L \text{ double indexed by } f} (\bigwedge_{j,x} (x \in f(j) \Rightarrow G(j)(x) = F(\langle j, x\rangle)) \wedge a = \mathrm{Inf}(\mathrm{Sups}(G))))\}$ holds $\mathrm{Inf}(\mathrm{Sups}(\mathrm{curry}\, F)) = \sup X$.

2. Completely-Distributive Lattices

Let L be a non empty relational structure. We say that L is completely-distributive if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) L is complete, and

(ii) for every non empty set J and for every non-empty many sorted set K indexed by J and for every set F of elements of L double indexed by K holds $\mathrm{Inf}(\mathrm{Sups}(F)) = \mathrm{Sup}(\mathrm{Infs}(\mathrm{Frege}(F)))$.

In the sequel J will denote a non empty set and K will denote a non-empty many sorted set indexed by J.

One can check that every non empty poset which is trivial is also completely-distributive.

One can verify that there exists a lattice which is completely-distributive.

Next we state the proposition

(24) Every completely-distributive lattice is continuous.

Let us observe that every lattice which is completely-distributive is also complete and continuous.

Next we state two propositions:

(25) Let L be a non empty antisymmetric transitive relational structure with g.l.b.'s, x be an element of L, and X, Y be subsets of L. Suppose $\sup X$ exists in L and $\sup Y$ exists in L and $Y = \{x \sqcap y, y$ ranges over elements of L: $y \in X\}$. Then $x \sqcap \sup X \geqslant \sup Y$.

(26) Let L be a completely-distributive lattice, X be a subset of L, and x be an element of L. Then $x \sqcap \sup X = \bigsqcup_L \{x \sqcap y, y$ ranges over elements of L: $y \in X\}$.

Let us note that every lattice which is completely-distributive is also Heyting.

3. Sub–Frames of Continuous Lattices

Let L be a non empty relational structure. A continuous subframe of L is an infs-inheriting directed-sups-inheriting non empty full relational substructure of L.

We now state three propositions:

(27) Let F be a set of elements of L double indexed by K. If for every element j of J holds $\operatorname{rng} F(j)$ is directed, then $\operatorname{rng} \operatorname{Infs}(\operatorname{Frege}(F))$ is directed.

(28) If L is continuous, then every continuous subframe of L is a continuous lattice.

(29) For every non empty poset S such that there exists a map from L into S which is infs-preserving and onto holds S is a complete lattice.

Let J be a set and let y be arbitrary. We introduce $J \Longmapsto y$ as a synonym of $J \longmapsto y$.

Let J be a set and let y be arbitrary. Then $J \longmapsto y$ is a many sorted set indexed by J. We introduce $J \Longmapsto y$ as a synonym of $J \longmapsto y$.

Let A, B, J be sets and let f be a function from A into B. Then $J \Longmapsto f$ is a many sorted function from $J \longmapsto A$ into $J \longmapsto B$.

We now state four propositions:

(30) Let A, B be sets, f be a function from A into B, and g be a function from B into A. If $g \cdot f = \mathrm{id}_A$, then $(J \Longmapsto g) \circ (J \Longmapsto f) = \mathrm{id}_{J \longmapsto A}$.

(31) Let J, A be non empty sets, B be a set, K be a many sorted set indexed by J, F be a set of elements of A double indexed by K, and f be a function from A into B. Then $(J \Longmapsto f) \circ F$ is a set of elements of B double indexed by K.

(32) Let J, A, B be non empty sets, K be a many sorted set indexed by J, F be a set of elements of A double indexed by K, and f be a function from A into B. Then $\mathrm{dom}_\kappa((J \Longmapsto f) \circ F)(\kappa) = \mathrm{dom}_\kappa F(\kappa)$.

(33) Suppose L is continuous. Let S be a non empty poset. Suppose there exists a map from L into S which is infs-preserving, directed-sups-preserving, and onto. Then S is a continuous lattice.

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Miscellaneous Facts about Relation Structure[1]

Agnieszka Julia Marasik
Warsaw University
Białystok

Summary. In the article notation and facts necessary to start with formalization of continuous lattices according to [5] are introduced.

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The papers [1], [3], [4], [2], [6], and [7] provide the terminology and notation for this paper.

1. Introduction

One can prove the following propositions:

(1) For every reflexive antisymmetric relational structure L with l.u.b.'s and for every element a of L holds $a \sqcup a = a$.

(2) For every reflexive antisymmetric relational structure L with g.l.b.'s and for every element a of L holds $a \sqcap a = a$.

(3) Let L be a transitive antisymmetric relational structure with l.u.b.'s and a, b, c be elements of L. If $a \sqcup b \leqslant c$, then $a \leqslant c$.

(4) Let L be a transitive antisymmetric relational structure with g.l.b.'s and a, b, c be elements of L. If $c \leqslant a \sqcap b$, then $c \leqslant a$.

(5) Let L be an antisymmetric transitive relational structure with l.u.b.'s and g.l.b.'s and a, b, c be elements of L. Then $a \sqcap b \leqslant a \sqcup c$.

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(6) Let L be an antisymmetric transitive relational structure with g.l.b.'s and a, b, c be elements of L. If $a \leqslant b$, then $a \sqcap c \leqslant b \sqcap c$.

(7) Let L be an antisymmetric transitive relational structure with l.u.b.'s and a, b, c be elements of L. If $a \leqslant b$, then $a \sqcup c \leqslant b \sqcup c$.

(8) For every sup-semilattice L and for all elements a, b of L such that $a \leqslant b$ holds $a \sqcup b = b$.

(9) For every sup-semilattice L and for all elements a, b, c of L such that $a \leqslant c$ and $b \leqslant c$ holds $a \sqcup b \leqslant c$.

(10) For every semilattice L and for all elements a, b of L such that $b \leqslant a$ holds $a \sqcap b = b$.

2. Difference in Relation Structure

We now state the proposition

(11) For every Boolean lattice L and for all elements x, y of L holds y is a complement of x iff $y = \neg x$.

Let L be a non empty relational structure and let a, b be elements of L. The functor $a \setminus b$ yielding an element of L is defined as follows:

(Def. 1) $a \setminus b = a \sqcap \neg b$.

Let L be a non empty relational structure and let a, b be elements of L. The functor $a \dot{-} b$ yields an element of L and is defined as follows:

(Def. 2) $a \dot{-} b = (a \setminus b) \sqcup (b \setminus a)$.

Let L be an antisymmetric relational structure with g.l.b.'s and l.u.b.'s and let a, b be elements of L. Let us notice that the functor $a \dot{-} b$ is commutative.

Let L be a non empty relational structure and let a, b be elements of L. We say that a meets b if and only if:

(Def. 3) $a \sqcap b \neq \bot_L$.

We introduce a misses b as an antonym of a meets b.

Let L be an antisymmetric relational structure with g.l.b.'s and let a, b be elements of L. Let us note that the predicate a meets b is symmetric. We introduce a misses b as an antonym of a meets b.

Next we state a number of propositions:

(12) Let L be an antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b, c be elements of L. If $a \leqslant c$, then $a \setminus b \leqslant c$.

(13) Let L be an antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b, c be elements of L. If $a \leqslant b$, then $a \setminus c \leqslant b \setminus c$.

(14) Let L be an antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b be elements of L. Then $a \setminus b \leqslant a$.

(15) Let L be an antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b be elements of L. Then $a \setminus b \leqslant a \dot{-} b$.

(16) For every lattice L and for all elements a, b, c of L such that $a \setminus b \leqslant c$ and $b \setminus a \leqslant c$ holds $a \dot{-} b \leqslant c$.

(17) For every lattice L and for every element a of L holds a meets a iff $a \neq \perp_L$.

(18) Let L be an antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b, c be elements of L. Then $a \sqcap (b \setminus c) = a \sqcap b \setminus c$.

(19) Let L be an antisymmetric transitive relational structure with g.l.b.'s. Suppose L is distributive. Let a, b, c be elements of L. If $a \sqcap b \sqcup a \sqcap c = a$, then $a \leqslant b \sqcup c$.

(20) For every lattice L such that L is distributive and for all elements a, b, c of L holds $a \sqcup b \sqcap c = (a \sqcup b) \sqcap (a \sqcup c)$.

(21) For every lattice L such that L is distributive and for all elements a, b, c of L holds $(a \sqcup b) \setminus c = (a \setminus c) \sqcup (b \setminus c)$.

3. Lower-bound in Relation Structure

Next we state a number of propositions:

(22) Let L be a lower-bounded non empty antisymmetric relational structure and a be an element of L. If $a \leqslant \perp_L$, then $a = \perp_L$.

(23) Let L be a lower-bounded semilattice and a, b, c be elements of L. If $a \leqslant b$ and $a \leqslant c$ and $b \sqcap c = \perp_L$, then $a = \perp_L$.

(24) Let L be a lower-bounded antisymmetric relational structure with l.u.b.'s and a, b be elements of L. If $a \sqcup b = \perp_L$, then $a = \perp_L$ and $b = \perp_L$.

(25) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and a, b, c be elements of L. If $a \leqslant b$ and $b \sqcap c = \perp_L$, then $a \sqcap c = \perp_L$.

(26) For every lower-bounded semilattice L and for every element a of L holds $\perp_L \setminus a = \perp_L$.

(27) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and a, b, c be elements of L. If a meets b and $b \leqslant c$, then a meets c.

(28) Let L be a lower-bounded antisymmetric relational structure with g.l.b.'s and a be an element of L. Then $a \sqcap \perp_L = \perp_L$.

(29) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b, c be elements of L. If a meets $b \sqcap c$, then a meets b.

(30) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b, c be elements of L. If a meets $b \setminus c$, then a meets b.

(31) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and a be an element of L. Then a misses $\perp_L$.

(32) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and a, b, c be elements of L. If a misses c and $b \leqslant c$, then a misses b.
(33) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and a, b, c be elements of L. If a misses b or a misses c, then a misses $b \sqcap c$.
(34) Let L be a lower-bounded lattice and a, b, c be elements of L. If $a \leqslant b$ and $a \leqslant c$ and b misses c, then $a = \bot_L$.
(35) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and a, b, c be elements of L. If a misses b, then $a \sqcap c$ misses $b \sqcap c$.

4. Boolean Lattices

We adopt the following rules: L will denote a Boolean non empty relational structure and a, b, c, d will denote elements of L.

Next we state a number of propositions:
(36) $a \sqcap b \sqcup b \sqcap c \sqcup c \sqcap a = (a \sqcup b) \sqcap (b \sqcup c) \sqcap (c \sqcup a)$.
(37) $a \sqcap \neg a = \bot_L$ and $a \sqcup \neg a = \top_L$.
(38) If $a \setminus b \leqslant c$, then $a \leqslant b \sqcup c$.
(39) $\neg(a \sqcup b) = \neg a \sqcap \neg b$ and $\neg(a \sqcap b) = \neg a \sqcup \neg b$.
(40) If $a \leqslant b$, then $\neg b \leqslant \neg a$.
(41) If $a \leqslant b$, then $c \setminus b \leqslant c \setminus a$.
(42) If $a \leqslant b$ and $c \leqslant d$, then $a \setminus d \leqslant b \setminus c$.
(43) If $a \leqslant b \sqcup c$, then $a \setminus b \leqslant c$ and $a \setminus c \leqslant b$.
(44) $\neg a \leqslant \neg(a \sqcap b)$ and $\neg b \leqslant \neg(a \sqcap b)$.
(45) $\neg(a \sqcup b) \leqslant \neg a$ and $\neg(a \sqcup b) \leqslant \neg b$.
(46) If $a \leqslant b \setminus a$, then $a = \bot_L$.
(47) If $a \leqslant b$, then $b = a \sqcup (b \setminus a)$.
(48) $a \setminus b = \bot_L$ iff $a \leqslant b$.
(49) If $a \leqslant b \sqcup c$ and $a \sqcap c = \bot_L$, then $a \leqslant b$.
(50) $a \sqcup b = (a \setminus b) \sqcup b$.
(51) $a \setminus (a \sqcup b) = \bot_L$.
(52) $a \setminus a \sqcap b = a \setminus b$.
(53) $(a \setminus b) \sqcap b = \bot_L$.
(54) $a \sqcup (b \setminus a) = a \sqcup b$.
(55) $a \sqcap b \sqcup (a \setminus b) = a$.
(56) $a \setminus (b \setminus c) = (a \setminus b) \sqcup a \sqcap c$.
(57) $a \setminus (a \setminus b) = a \sqcap b$.

(58) $(a \sqcup b) \setminus b = a \setminus b$.

(59) $a \sqcap b = \bot_L$ iff $a \setminus b = a$.

(60) $a \setminus (b \sqcup c) = (a \setminus b) \sqcap (a \setminus c)$.

(61) $a \setminus b \sqcap c = (a \setminus b) \sqcup (a \setminus c)$.

(62) $a \sqcap (b \setminus c) = a \sqcap b \setminus a \sqcap c$.

(63) $(a \sqcup b) \setminus a \sqcap b = (a \setminus b) \sqcup (b \setminus a)$.

(64) $a \setminus b \setminus c = a \setminus (b \sqcup c)$.

(65) $\neg(\bot_L) = \top_L$.

(66) $\neg(\top_L) = \bot_L$.

(67) $a \setminus a = \bot_L$.

(68) $a \setminus \bot_L = a$.

(69) $\neg(a \setminus b) = \neg a \sqcup b$.

(70) $a \sqcap b$ misses $a \setminus b$.

(71) $a \setminus b$ misses b.

(72) If a misses b, then $(a \sqcup b) \setminus b = a$.

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Moore-Smith Convergence[1]

Andrzej Trybulec
Warsaw University
Białystok

Summary. The paper introduces the concept of a net (a generalized sequence). The goal is to enable the continuation of the translation of [16].

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The notation and terminology used here are introduced in the following papers: [30], [36], [35], [13], [31], [14], [37], [38], [11], [12], [10], [26], [9], [1], [2], [33], [23], [24], [3], [4], [25], [18], [20], [39], [15], [27], [32], [21], [34], [5], [28], [6], [7], [17], [19], [29], [8], and [22].

1. Preliminaries

The scheme *SubsetEq* deals with a non empty set $\mathcal{A}$, subsets $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{B} = \mathcal{C}$

provided the following conditions are met:

- For every element y of $\mathcal{A}$ holds $y \in \mathcal{B}$ iff $\mathcal{P}[y]$,
- For every element y of $\mathcal{A}$ holds $y \in \mathcal{C}$ iff $\mathcal{P}[y]$.

We now state the proposition

(1) For all sets X, x holds $X \longmapsto x$ is constant.

Let X, x be sets. Note that $X \longmapsto x$ is constant.

Let f be a function. Let us assume that f is non empty and constant. The value of f is defined by:

(Def. 1) There exists a set x such that $x \in \operatorname{dom} f$ and the value of $f = f(x)$.

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Let us note that there exists a function which is non empty and constant.

Let f be a non empty constant function. Then the value of f can be characterized by the condition:

(Def. 2) There exists a set x such that $x \in \operatorname{dom} f$ and the value of $f = f(x)$.

The following propositions are true:

(2) For every non empty set X and for every set x holds the value of $X \longmapsto x = x$.

(3) For every function f holds $\overline{\overline{\operatorname{rng} f}} \subseteq \overline{\overline{\operatorname{dom} f}}$.

Let us note that every set which is universal is also transitive and a Tarski class and every set which is transitive and a Tarski class is also universal.

In the sequel x, X will be sets and T will be a universal class.

Let us consider X. The universe of X is defined as follows:

(Def. 3) The universe of $X = \mathbf{T}(X^{*\in})$.

We now state the proposition

(4) $\mathbf{T}(X)$ is a Tarski class.

Let us consider X. Note that $\mathbf{T}(X)$ is a Tarski class.

Let us consider X. Observe that the universe of X is transitive and a Tarski class.

Let us consider X. One can check that the universe of X is universal and non empty.

One can prove the following proposition

(5) For every function f such that $\operatorname{dom} f \in T$ and $\operatorname{rng} f \subseteq T$ holds $\prod f \in T$.

2. Topological spaces

Next we state the proposition

(6) Let T be a non empty topological space, A be a subset of T, and p be a point of T. Then $p \in \overline{A}$ if and only if for every neighbourhood G of p holds G meets A.

Let T be a non empty topological space. We introduce T is Hausdorff as a synonym of T is T_2.

One can verify that there exists a non empty topological space which is Hausdorff.

One can prove the following two propositions:

(7) Let X be a non empty topological space and A be a subset of the carrier of X. Then Ω_X is a neighbourhood of A.

(8) Let X be a non empty topological space, A be a subset of the carrier of X, and Y be a neighbourhood of A. Then $A \subseteq Y$.

3. 1-sorted structures

The following proposition is true

(9) Let Y be a non empty set, J be a 1-sorted yielding many sorted set indexed by Y, and i be an element of Y. Then $(\operatorname{support} J)(i) =$ the carrier of $J(i)$.

Let us note that there exists a function which is non empty, constant, and 1-sorted yielding.

Let J be a 1-sorted yielding function. Let us observe that J is nonempty if and only if:

(Def. 4) For every set i such that $i \in \operatorname{rng} J$ holds i is a non empty 1-sorted structure.

We introduce J is yielding non-empty carriers as a synonym of J is nonempty.

Let X be a set and let L be a 1-sorted structure. Observe that $X \longmapsto L$ is 1-sorted yielding.

Let I be a set. Observe that there exists a 1-sorted yielding many sorted set indexed by I which is yielding non-empty carriers.

Let I be a non empty set and let J be a relational structure yielding many sorted set indexed by I. One can verify that the carrier of $\prod J$ is functional.

Let I be a set and let J be a yielding non-empty carriers 1-sorted yielding many sorted set indexed by I. Observe that support J is non-empty.

Next we state the proposition

(10) Let T be a non empty 1-sorted structure, S be a subset of the carrier of T, and p be an element of the carrier of T. Then $p \notin S$ if and only if $p \in -S$.

4. Relational structures

Let T be a non empty relational structure and let A be a lower subset of T. Observe that $-A$ is upper.

Let T be a non empty relational structure and let A be an upper subset of T. Observe that $-A$ is lower.

Let N be a non empty relational structure. Let us observe that N is directed if and only if:

(Def. 5) For all elements x, y of N there exists an element z of N such that $x \leqslant z$ and $y \leqslant z$.

Let X be a set. Note that $2^X_\subseteq$ is directed.

Let us mention that there exists a relational structure which is non empty, directed, transitive, and strict.

Let M be a non empty set, let N be a non empty relational structure, let f be a function from M into the carrier of N, and let m be an element of M. Then $f(m)$ is an element of N.

Let I be a set. Note that there exists a relational structure yielding many sorted set indexed by I which is yielding non-empty carriers.

Let I be a non empty set and let J be a yielding non-empty carriers relational structure yielding many sorted set indexed by I. Observe that $\prod J$ is non empty.

Next we state the proposition

(11) For all relational structures R_1, R_2 holds $\Omega_{[: R_1, R_2 :]} = [: \Omega_{(R_1)}, \Omega_{(R_2)} :]$.

Let Y_1, Y_2 be directed relational structures. Observe that $[: Y_1, Y_2 :]$ is directed.

Next we state the proposition

(12) For every relational structure R holds the carrier of R = the carrier of $R^{\smile}$.

Let S be a 1-sorted structure and let N be a net structure over S. We say that N is constant if and only if:

(Def. 6) The mapping of N is constant.

Let R be a relational structure, let T be a non empty 1-sorted structure, and let p be an element of the carrier of T. The functor $R \longmapsto p$ yielding a strict net structure over T is defined by the conditions (Def. 7).

(Def. 7)(i) The relational structure of $(R \longmapsto p)$ = the relational structure of R, and

(ii) the mapping of $(R \longmapsto p)$ = (the carrier of $(R \longmapsto p)) \longmapsto p$.

Let R be a relational structure, let T be a non empty 1-sorted structure, and let p be an element of the carrier of T. Note that $R \longmapsto p$ is constant.

Let R be a non empty relational structure, let T be a non empty 1-sorted structure, and let p be an element of the carrier of T. One can verify that $R \longmapsto p$ is non empty.

Let R be a non empty directed relational structure, let T be a non empty 1-sorted structure, and let p be an element of the carrier of T. Note that $R \longmapsto p$ is directed.

Let R be a non empty transitive relational structure, let T be a non empty 1-sorted structure, and let p be an element of the carrier of T. One can check that $R \longmapsto p$ is transitive.

We now state two propositions:

(13) Let R be a relational structure, T be a non empty 1-sorted structure, and p be an element of the carrier of T. Then the carrier of $(R \longmapsto p)$ = the carrier of R.

(14) Let R be a non empty relational structure, T be a non empty 1-sorted structure, p be an element of the carrier of T, and q be an element of the carrier of $(R \longmapsto p)$.Then $(R \longmapsto p)(q) = p$.

Let T be a non empty 1-sorted structure and let N be a non empty net structure over T. Observe that the mapping of N is non empty.

5. Substructures of nets

One can prove the following propositions:

(15) Every relational structure R is a full relational substructure of R.

(16) Let R be a relational structure and S be a relational substructure of R. Then every relational substructure of S is a relational substructure of R.

Let S be a 1-sorted structure and let N be a net structure over S. A net structure over S is called a structure of a subnet of N if:

(Def. 8) It is a relational substructure of N and the mapping of it $=$ (the mapping of N)$\restriction$(the carrier of it).

Next we state two propositions:

(17) For every 1-sorted structure S holds every net structure N over S is a structure of a subnet of N.

(18) Let Q be a 1-sorted structure, R be a net structure over Q, and S be a structure of a subnet of R. Then every structure of a subnet of S is a structure of a subnet of R.

Let S be a 1-sorted structure, let N be a net structure over S, and let M be a structure of a subnet of N. We say that M is full if and only if:

(Def. 9) M is a full relational substructure of N.

Let S be a 1-sorted structure and let N be a net structure over S. Note that there exists a structure of a subnet of N which is full and strict.

Let S be a 1-sorted structure and let N be a non empty net structure over S. Note that there exists a structure of a subnet of N which is full, non empty, and strict.

One can prove the following three propositions:

(19) Let S be a 1-sorted structure, N be a net structure over S, and M be a structure of a subnet of N. Then the carrier of $M \subseteq$ the carrier of N.

(20) Let S be a 1-sorted structure, N be a net structure over S, M be a structure of a subnet of N, x, y be elements of N, and i, j be elements of the carrier of M. If $x = i$ and $y = j$ and $i \leqslant j$, then $x \leqslant y$.

(21) Let S be a 1-sorted structure, N be a non empty net structure over S, M be a non empty full structure of a subnet of N, x, y be elements of N, and i, j be elements of the carrier of M. If $x = i$ and $y = j$ and $x \leqslant y$, then $i \leqslant j$.

6. More about nets

Let T be a non empty 1-sorted structure. One can verify that there exists a net in T which is constant and strict.

Let T be a non empty 1-sorted structure and let N be a constant net structure over T. One can verify that the mapping of N is constant.

Let T be a non empty 1-sorted structure and let N be a net structure over T. Let us assume that N is constant and non empty. The value of N yields an element of T and is defined as follows:

(Def. 10) The value of $N =$ the value of the mapping of N.

Let T be a non empty 1-sorted structure and let N be a constant non empty net structure over T. Then the value of N can be characterized by the condition:

(Def. 11) The value of $N =$ the value of the mapping of N.

Next we state the proposition

(22) Let R be a non empty relational structure, T be a non empty 1-sorted structure, and p be an element of the carrier of T. Then the value of $R \longmapsto p = p$.

Let T be a non empty 1-sorted structure and let N be a net in T. A net in T is said to be a subnet of N if it satisfies the condition (Def. 12).

(Def. 12) There exists a map f from it into N such that
  (i) the mapping of it $=$ (the mapping of N) $\cdot f$, and
  (ii) for every element m of N there exists an element n of it such that for every element p of it such that $n \leqslant p$ holds $m \leqslant f(p)$.

We now state several propositions:

(23) For every non empty 1-sorted structure T holds every net N in T is a subnet of N.

(24) Let T be a non empty 1-sorted structure and N_1, N_2, N_3 be nets in T. Suppose N_1 is a subnet of N_2 and N_2 is a subnet of N_3. Then N_1 is a subnet of N_3.

(25) Let T be a non empty 1-sorted structure, N be a constant net in T, and i be an element of the carrier of N. Then $N(i) =$ the value of N.

(26) Let L be a non empty 1-sorted structure, N be a net in L, and X, Y be sets. If N is eventually in X and eventually in Y, then X meets Y.

(27) Let S be a non empty 1-sorted structure, N be a net in S, M be a subnet of N, and given X. If M is often in X, then N is often in X.

(28) Let S be a non empty 1-sorted structure, N be a net in S, and given X. If N is eventually in X, then N is often in X.

(29) For every non empty 1-sorted structure S holds every net in S is eventually in the carrier of S.

7. The restriction of a net

Let S be a 1-sorted structure, let N be a net structure over S, and let us consider X. The functor $N^{-1}(X)$ yields a strict structure of a subnet of N and is defined by:

(Def. 13) $N^{-1}(X)$ is a full relational substructure of N and the carrier of $N^{-1}(X) =$ (the mapping of $N)^{-1}(X)$.

Let S be a 1-sorted structure, let N be a transitive net structure over S, and let us consider X. One can verify that $N^{-1}(X)$ is transitive and full.

We now state three propositions:

(30) Let S be a non empty 1-sorted structure, N be a net in S, and given X. If N is often in X, then $N^{-1}(X)$ is non empty and directed.

(31) Let S be a non empty 1-sorted structure, N be a net in S, and given X. If N is often in X, then $N^{-1}(X)$ is a subnet of N.

(32) Let S be a non empty 1-sorted structure, N be a net in S, given X, and M be a subnet of N. If $M = N^{-1}(X)$, then M is eventually in X.

8. The universe of nets

Let X be a non empty 1-sorted structure. The functor $\mathrm{NetUniv}(X)$ is defined by the condition (Def. 14).

(Def. 14) Let given x. Then $x \in \mathrm{NetUniv}(X)$ if and only if there exists a strict net N in X such that $N = x$ and the carrier of $N \in$ the universe of the carrier of X.

Let X be a non empty 1-sorted structure. One can check that $\mathrm{NetUniv}(X)$ is non empty.

9. Parametrized families of nets, iteration

Let X be a set and let T be a 1-sorted structure. A many sorted set indexed by X is said to be a net set of X, T if:

(Def. 15) For every set i such that $i \in \mathrm{rng}\,\mathrm{it}$ holds i is a net in T.

The following proposition is true

(33) Let X be a set, T be a 1-sorted structure, and F be a many sorted set indexed by X. Then F is a net set of X, T if and only if for every set i such that $i \in X$ holds $F(i)$ is a net in T.

Let X be a non empty set, let T be a 1-sorted structure, let J be a net set of X, T, and let i be an element of X. Then $J(i)$ is a net in T.

Let X be a set and let T be a 1-sorted structure. One can check that every net set of X, T is relational structure yielding.

Let T be a 1-sorted structure and let Y be a net in T. Observe that every net set of the carrier of Y, T is yielding non-empty carriers.

Let T be a non empty 1-sorted structure, let Y be a net in T, and let J be a net set of the carrier of Y, T. One can check that $\prod J$ is directed and transitive.

Let X be a set and let T be a 1-sorted structure. Observe that every net set of X, T is yielding non-empty carriers.

Let X be a set and let T be a 1-sorted structure. One can check that there exists a net set of X, T which is yielding non-empty carriers.

Let T be a non empty 1-sorted structure, let Y be a net in T, and let J be a net set of the carrier of Y, T. The functor Iterated(J) yielding a strict net in T is defined by the conditions (Def. 16).

(Def. 16)(i) The relational structure of Iterated$(J) = [\!:Y, \prod J:\!]$, and

(ii) for every element i of the carrier of Y and for every function f such that $i \in$ the carrier of Y and $f \in$ the carrier of $\prod J$ holds (the mapping of Iterated$(J))(i, f) =$ (the mapping of $J(i))(f(i))$.

We now state four propositions:

(34) Let T be a non empty 1-sorted structure, Y be a net in T, and J be a net set of the carrier of Y, T. Suppose $Y \in$ NetUniv(T) and for every element i of the carrier of Y holds $J(i) \in$ NetUniv(T). Then Iterated$(J) \in$ NetUniv(T).

(35) Let T be a non empty 1-sorted structure, N be a net in T, and J be a net set of the carrier of N, T. Then the carrier of Iterated$(J) = [\!:$the carrier of N, $\prod$support $J:\!]$.

(36) Let T be a non empty 1-sorted structure, N be a net in T, J be a net set of the carrier of N, T, i be an element of the carrier of N, f be an element of the carrier of $\prod J$, and x be an element of the carrier of Iterated(J). If $x = \langle i, f\rangle$, then (Iterated$(J))(x) =$ (the mapping of $J(i))(f(i))$.

(37) Let T be a non empty 1-sorted structure, Y be a net in T, and J be a net set of the carrier of Y, T. Then rng (the mapping of Iterated$(J)) \subseteq \bigcup\{$rng (the mapping of $J(i))$: i ranges over elements of $Y\}$.

10. Poset of open neighbourhoods

Let T be a non empty topological space and let p be a point of T. The open neighbourhoods of p constitute a relational structure and is defined as follows:

(Def. 17) The open neighbourhoods of $p = (\langle\{V, V$ ranges over subsets of T: $p \in V \wedge V$ is open$\}, \subseteq\rangle)^{\smile}$.

Let T be a non empty topological space and let p be a point of T. One can check that the open neighbourhoods of p is non empty.

One can prove the following propositions:

(38) Let T be a non empty topological space, p be a point of T, and x be an element of the carrier of the open neighbourhoods of p. Then there exists a subset W of T such that $W = x$ and $p \in W$ and W is open.

(39) Let T be a non empty topological space, p be a point of T, and x be a subset of the carrier of T. Then $x \in$ the carrier of the open neighbourhoods of p if and only if $p \in x$ and x is open.

(40) Let T be a non empty topological space, p be a point of T, and x, y be elements of the carrier of the open neighbourhoods of p. Then $x \leqslant y$ if and only if $y \subseteq x$.

Let T be a non empty topological space and let p be a point of T. Note that the open neighbourhoods of p is transitive and directed.

11. Nets in topological spaces

Let T be a non empty topological space and let N be a net in T. The functor $\mathrm{Lim}\, N$ yields a subset of T and is defined as follows:

(Def. 18) For every point p of T holds $p \in \mathrm{Lim}\, N$ iff for every neighbourhood V of p holds N is eventually in V.

The following four propositions are true:

(41) For every non empty topological space T and for every net N in T and for every subnet Y of N holds $\mathrm{Lim}\, N \subseteq \mathrm{Lim}\, Y$.

(42) For every non empty topological space T and for every constant net N in T holds the value of $N \in \mathrm{Lim}\, N$.

(43) Let T be a non empty topological space, N be a net in T, and p be a point of T. Suppose $p \in \mathrm{Lim}\, N$. Let d be an element of N. Then there exists a subset S of T such that $S = \{N(c), c$ ranges over elements of N: $d \leqslant c\}$ and $p \in \overline{S}$.

(44) Let T be a non empty topological space. Then T is Hausdorff if and only if for every net N in T and for all points p, q of T such that $p \in \mathrm{Lim}\, N$ and $q \in \mathrm{Lim}\, N$ holds $p = q$.

Let T be a Hausdorff non empty topological space and let N be a net in T. Observe that $\mathrm{Lim}\, N$ is trivial.

Let T be a non empty topological space and let N be a net in T. We say that N is convergent if and only if:

(Def. 19) $\mathrm{Lim}\, N \neq \emptyset$.

Let T be a non empty topological space. Observe that every net in T which is constant is also convergent.

Let T be a non empty topological space. Note that there exists a net in T which is convergent and strict.

Let T be a Hausdorff non empty topological space and let N be a convergent net in T. The functor $\lim N$ yielding an element of T is defined as follows:

(Def. 20) $\lim N \in \mathrm{Lim}\, N$.

One can prove the following propositions:

(45) For every Hausdorff non empty topological space T and for every constant net N in T holds $\lim N =$ the value of N.

(46) Let T be a non empty topological space, N be a net in T, and p be a point of T. Suppose $p \notin \mathrm{Lim}\, N$. Then it is not true that there exists a subnet Y of N and there exists a subnet Z of Y such that $p \in \mathrm{Lim}\, Z$.

(47) Let T be a non empty topological space and N be a net in T. Suppose $N \in \mathrm{NetUniv}(T)$. Let p be a point of T. Suppose $p \notin \mathrm{Lim}\, N$. Then there exists a subnet Y of N such that $Y \in \mathrm{NetUniv}(T)$ and it is not true that there exists a subnet Z of Y such that $p \in \mathrm{Lim}\, Z$.

(48) Let T be a non empty topological space, N be a net in T, and p be a point of T. Suppose $p \in \mathrm{Lim}\, N$. Let J be a net set of the carrier of N, T. Suppose that for every element i of the carrier of N holds $N(i) \in \mathrm{Lim}\, J(i)$. Then $p \in \mathrm{Lim}\,\mathrm{Iterated}(J)$.

12. Convergence classes

Let S be a non empty 1-sorted structure. Convergence class of S is defined as follows:

(Def. 21) It $\subseteq$ [: NetUniv(S), the carrier of S :].

Let S be a non empty 1-sorted structure. Note that every convergence class of S is relation-like.

Let T be a non empty topological space. The functor Convergence(T) yielding a convergence class of T is defined as follows:

(Def. 22) For every net N in T and for every point p of T holds $\langle N, p\rangle \in \mathrm{Convergence}(T)$ iff $N \in \mathrm{NetUniv}(T)$ and $p \in \mathrm{Lim}\, N$.

Let T be a non empty 1-sorted structure and let C be a convergence class of T. We say that C has (CONSTANTS) property if and only if:

(Def. 23) For every constant net N in T such that $N \in \mathrm{NetUniv}(T)$ holds $\langle N$, the value of $N\rangle \in C$.

We say that C has (SUBNETS) property if and only if the condition (Def. 24) is satisfied.

(Def. 24) Let N be a net in T and Y be a subnet of N. Suppose $Y \in \mathrm{NetUniv}(T)$. Let p be an element of the carrier of T. If $\langle N, p\rangle \in C$, then $\langle Y, p\rangle \in C$.

We say that C has (DIVERGENCE) property if and only if the condition (Def. 25) is satisfied.

(Def. 25) Let X be a net in T and p be an element of the carrier of T. Suppose $X \in \mathrm{NetUniv}(T)$ and $\langle X, p\rangle \notin C$. Then there exists a subnet Y of X such that $Y \in \mathrm{NetUniv}(T)$ and it is not true that there exists a subnet Z of Y such that $\langle Z, p\rangle \in C$.

We say that C has (ITERATED LIMITS) property if and only if the condition (Def. 26) is satisfied.

(Def. 26) Let X be a net in T and p be an element of the carrier of T. Suppose $\langle X, p\rangle \in C$. Let J be a net set of the carrier of X, T. Suppose that for every element i of the carrier of X holds $\langle J(i), X(i)\rangle \in C$. Then $\langle \mathrm{Iterated}(J), p\rangle \in C$.

Let T be a non empty topological space. Note that Convergence(T) has (CONSTANTS) property, (SUBNETS) property, (DIVERGENCE) property, and (ITERATED LIMITS) property.

Let S be a non empty 1-sorted structure and let C be a convergence class of S. The functor ConvergenceSpace(C) yielding a strict topological structure is defined by the conditions (Def. 27).

(Def. 27)(i) The carrier of ConvergenceSpace(C) = the carrier of S, and

(ii) the topology of ConvergenceSpace(C) = $\{V, V$ ranges over subsets of the carrier of S: $\bigwedge_{p\,:\,\text{element of the carrier of } S}$ $(p \in V \;\Rightarrow\; \bigwedge_{N\,:\,\text{net in } S}$ $(\langle N, p\rangle \in C \;\Rightarrow\; N$ is eventually in $V))\}$.

Let S be a non empty 1-sorted structure and let C be a convergence class of S. Observe that ConvergenceSpace(C) is non empty.

Let S be a non empty 1-sorted structure and let C be a convergence class of S. Note that ConvergenceSpace(C) is topological space-like.

One can prove the following proposition

(49) For every non empty 1-sorted structure S and for every convergence class C of S holds $C \subseteq$ Convergence(ConvergenceSpace(C)).

Let T be a non empty 1-sorted structure and let C be a convergence class of T. We say that C is topological if and only if:

(Def. 28) C has (CONSTANTS) property, (SUBNETS) property, (DIVERGENCE) property, and (ITERATED LIMITS) property.

Let T be a non empty 1-sorted structure. One can check that there exists a convergence class of T which is non empty and topological.

Let T be a non empty 1-sorted structure. One can verify that every convergence class of T which is topological has (CONSTANTS) property, (SUBNETS) property, (DIVERGENCE) property, and (ITERATED LIMITS) property and every convergence class of T which has (CONSTANTS) property, (SUBNETS) property, (DIVERGENCE) property, and (ITERATED LIMITS) property is topological.

The following propositions are true:

(50) Let T be a non empty 1-sorted structure, C be a topological convergence class of T, and S be a subset of ConvergenceSpace(C) **qua** non empty topological space. Then S is open if and only if for every element p of the carrier of T such that $p \in S$ and for every net N in T such that $\langle N, p\rangle \in C$ holds N is eventually in S.

(51) Let T be a non empty 1-sorted structure, C be a topological convergence class of T, and S be a subset of ConvergenceSpace(C) **qua** non empty topological space. Then S is closed if and only if for every element p of the carrier of T and for every net N in T such that $\langle N, p\rangle \in C$ and N is often in S holds $p \in S$.

(52) Let T be a non empty 1-sorted structure, C be a topological convergence class of T, S be a subset of ConvergenceSpace(C), and p be a point of ConvergenceSpace(C). Suppose $p \in \overline{S}$. Then there exists a net N in

ConvergenceSpace(C) such that $\langle N, p\rangle \in C$ and rng (the mapping of N) $\subseteq S$ and $p \in \mathrm{Lim}\, N$.

(53) Let T be a non empty 1-sorted structure and C be a convergence class of T. Then Convergence(ConvergenceSpace(C)) $= C$ if and only if C is topological.

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Duality in Relation Structures[1]

Grzegorz Bancerek
Warsaw University
Białystok

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The articles [15], [18], [19], [21], [20], [7], [8], [10], [1], [2], [6], [14], [11], [16], [12], [17], [3], [4], [23], [9], [5], [22], and [13] provide the terminology and notation for this paper.

Let L be a relational structure. We introduce L^{op} as a synonym of $L^{\smile}$.

We now state several propositions:

(1) For every relational structure L and for all elements x, y of L^{op} holds $x \leqslant y$ iff $\curvearrowleft x \geqslant \curvearrowleft y$.

(2) Let L be a relational structure, x be an element of L, and y be an element of L^{op}. Then

(i) $x \leqslant \curvearrowleft y$ iff $x^{\smile} \geqslant y$, and

(ii) $x \geqslant \curvearrowleft y$ iff $x^{\smile} \leqslant y$.

(3) For every relational structure L holds L is empty iff L^{op} is empty.

(4) For every relational structure L holds L is reflexive iff L^{op} is reflexive.

(5) For every relational structure L holds L is antisymmetric iff L^{op} is antisymmetric.

(6) For every relational structure L holds L is transitive iff L^{op} is transitive.

(7) For every non empty relational structure L holds L is connected iff L^{op} is connected.

Let L be a reflexive relational structure. One can check that L^{op} is reflexive.

Let L be a transitive relational structure. One can check that L^{op} is transitive.

Let L be an antisymmetric relational structure. Note that L^{op} is antisymmetric.

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Let L be a connected non empty relational structure. Observe that $L^{\rm op}$ is connected.

One can prove the following propositions:

(8) Let L be a relational structure, x be an element of L, and X be a set. Then

(i) $x \leqslant X$ iff $x^{\smile} \geqslant X$, and

(ii) $x \geqslant X$ iff $x^{\smile} \leqslant X$.

(9) Let L be a relational structure, x be an element of $L^{\rm op}$, and X be a set. Then

(i) $x \leqslant X$ iff $\frown x \geqslant X$, and

(ii) $x \geqslant X$ iff $\frown x \leqslant X$.

(10) Let L be a relational structure and X be a set. Then sup X exists in L if and only if inf X exists in $L^{\rm op}$.

(11) Let L be a relational structure and X be a set. Then sup X exists in $L^{\rm op}$ if and only if inf X exists in L.

(12) Let L be a non empty relational structure and X be a set. If sup X exists in L or inf X exists in $L^{\rm op}$, then $\bigsqcup_L X = \bigsqcap_{(L^{\rm op})} X$.

(13) Let L be a non empty relational structure and X be a set. If inf X exists in L or sup X exists in $L^{\rm op}$, then $\bigsqcap_L X = \bigsqcup_{(L^{\rm op})} X$.

(14) For all relational structures L_1, L_2 such that the relational structure of L_1 = the relational structure of L_2 and L_1 has g.l.b.'s holds L_2 has g.l.b.'s.

(15) For all relational structures L_1, L_2 such that the relational structure of L_1 = the relational structure of L_2 and L_1 has l.u.b.'s holds L_2 has l.u.b.'s.

(16) For every relational structure L holds L has g.l.b.'s iff $L^{\rm op}$ has l.u.b.'s.

(17) For every non empty relational structure L holds L is complete iff $L^{\rm op}$ is complete.

Let L be a relational structure with g.l.b.'s. Note that $L^{\rm op}$ has l.u.b.'s.

Let L be a relational structure with l.u.b.'s. One can check that $L^{\rm op}$ has g.l.b.'s.

Let L be a complete non empty relational structure. One can check that $L^{\rm op}$ is complete.

The following propositions are true:

(18) Let L be a non empty relational structure, X be a subset of L, and Y be a subset of $L^{\rm op}$. If $X = Y$, then fininfs(X) = finsups(Y) and finsups(X) = fininfs(Y).

(19) Let L be a relational structure, X be a subset of L, and Y be a subset of $L^{\rm op}$. If $X = Y$, then $\downarrow X = \uparrow Y$ and $\uparrow X = \downarrow Y$.

(20) Let L be a non empty relational structure, x be an element of L, and y be an element of $L^{\rm op}$. If $x = y$, then $\downarrow x = \uparrow y$ and $\uparrow x = \downarrow y$.

(21) For every poset L with g.l.b.'s and for all elements x, y of L holds $x \sqcap y = x^{\smile} \sqcup y^{\smile}$.

(22) For every poset L with g.l.b.'s and for all elements x, y of L^{op} holds $\curvearrowleft x \sqcap \curvearrowleft y = x \sqcup y$.

(23) For every poset L with l.u.b.'s and for all elements x, y of L holds $x \sqcup y = x^{\smile} \sqcap y^{\smile}$.

(24) For every poset L with l.u.b.'s and for all elements x, y of L^{op} holds $\curvearrowleft x \sqcup \curvearrowleft y = x \sqcap y$.

(25) For every lattice L holds L is distributive iff L^{op} is distributive.

Let L be a distributive lattice. One can check that L^{op} is distributive.

Next we state a number of propositions:

(26) Let L be a relational structure and x be a set. Then x is a directed subset of L if and only if x is a filtered subset of L^{op}.

(27) Let L be a relational structure and x be a set. Then x is a directed subset of L^{op} if and only if x is a filtered subset of L.

(28) Let L be a relational structure and x be a set. Then x is a lower subset of L if and only if x is an upper subset of L^{op}.

(29) Let L be a relational structure and x be a set. Then x is a lower subset of L^{op} if and only if x is an upper subset of L.

(30) For every relational structure L holds L is lower-bounded iff L^{op} is upper-bounded.

(31) For every relational structure L holds L^{op} is lower-bounded iff L is upper-bounded.

(32) For every relational structure L holds L is bounded iff L^{op} is bounded.

(33) For every lower-bounded antisymmetric non empty relational structure L holds $(\perp_L)^{\smile} = \top_{L^{\mathrm{op}}}$ and $\curvearrowleft(\top_{L^{\mathrm{op}}}) = \perp_L$.

(34) For every upper-bounded antisymmetric non empty relational structure L holds $(\top_L)^{\smile} = \perp_{L^{\mathrm{op}}}$ and $\curvearrowleft(\perp_{L^{\mathrm{op}}}) = \top_L$.

(35) Let L be a bounded lattice and x, y be elements of L. Then y is a complement of x if and only if $y^{\smile}$ is a complement of $x^{\smile}$.

(36) For every bounded lattice L holds L is complemented iff L^{op} is complemented.

Let L be a lower-bounded relational structure. One can verify that L^{op} is upper-bounded.

Let L be an upper-bounded relational structure. Note that L^{op} is lower-bounded.

Let L be a complemented bounded lattice. One can check that L^{op} is complemented.

Next we state the proposition

(37) For every Boolean lattice L and for every element x of L holds $\neg(x^{\smile}) = \neg x$.

Let L be a non empty relational structure. The functor $\neg_L$ yields a map from L into L^{op} and is defined as follows:

(Def. 1) For every element x of L holds $\neg_L(x) = \neg x$.

Let L be a Boolean lattice. Observe that $\neg_L$ is one-to-one.

Let L be a Boolean lattice. One can verify that $\neg_L$ is isomorphic.

The following propositions are true:

(38) For every Boolean lattice L holds L and $L^{\rm op}$ are isomorphic.

(39) Let S, T be non empty relational structures and f be a set. Then
 (i) f is a map from S into T iff f is a map from $S^{\rm op}$ into T,
 (ii) f is a map from S into T iff f is a map from S into $T^{\rm op}$, and
 (iii) f is a map from S into T iff f is a map from $S^{\rm op}$ into $T^{\rm op}$.

(40) Let S, T be non empty relational structures, f be a map from S into T, and g be a map from S into $T^{\rm op}$ such that $f = g$. Then
 (i) f is monotone iff g is antitone, and
 (ii) f is antitone iff g is monotone.

(41) Let S, T be non empty relational structures, f be a map from S into $T^{\rm op}$, and g be a map from $S^{\rm op}$ into T such that $f = g$. Then
 (i) f is monotone iff g is monotone, and
 (ii) f is antitone iff g is antitone.

(42) Let S, T be non empty relational structures, f be a map from S into T, and g be a map from $S^{\rm op}$ into $T^{\rm op}$ such that $f = g$. Then
 (i) f is monotone iff g is monotone, and
 (ii) f is antitone iff g is antitone.

(43) Let S, T be non empty relational structures and f be a set. Then
 (i) f is a connection between S and T iff f is a connection between $S^\smile$ and T,
 (ii) f is a connection between S and T iff f is a connection between S and $T^\smile$, and
 (iii) f is a connection between S and T iff f is a connection between $S^\smile$ and $T^\smile$.

(44) Let S, T be non empty posets, f_1 be a map from S into T, g_1 be a map from T into S, f_2 be a map from $S^\smile$ into $T^\smile$, and g_2 be a map from $T^\smile$ into $S^\smile$. If $f_1 = f_2$ and $g_1 = g_2$, then $\langle f_1, g_1\rangle$ is Galois iff $\langle g_2, f_2\rangle$ is Galois.

(45) Let J be a set, D be a non empty set, K be a many sorted set indexed by J, and F be a set of elements of D double indexed by K. Then $\mathrm{dom}_\kappa\, F(\kappa) = K$.

Let J, D be non empty sets, let K be a non-empty many sorted set indexed by J, let F be a set of elements of D double indexed by K, let j be an element of J, and let k be an element of $K(j)$. Then $F(j)(k)$ is an element of D.

One can prove the following propositions:

(46) Let L be a non empty relational structure, J be a set, K be a many sorted set indexed by J, and x be a set. Then x is a set of elements of L double indexed by K if and only if x is a set of elements of $L^{\rm op}$ double indexed by K.

(47) Let L be a complete lattice, J be a non empty set, K be a non-empty many sorted set indexed by J, and F be a set of elements of L double indexed by K. Then $\mathrm{Sup}(\mathrm{Infs}(F)) \leqslant \mathrm{Inf}(\mathrm{Sups}(\mathrm{Frege}(F)))$.

(48) Let L be a complete lattice. Then L is completely-distributive if and only if for every non empty set J and for every non-empty many sorted set K indexed by J and for every set F of elements of L double indexed by K holds $\mathrm{Sup}(\mathrm{Infs}(F)) = \mathrm{Inf}(\mathrm{Sups}(\mathrm{Frege}(F)))$.

(49) Let L be a complete antisymmetric non empty relational structure and F be a function. Then $\bigsqcup_L F = \bigsqcap_{(L^{\mathrm{op}})} F$ and $\bigsqcap_L F = \bigsqcup_{(L^{\mathrm{op}})} F$.

(50) Let L be a complete antisymmetric non empty relational structure and F be a function yielding function. Then $\underline{\bigsqcup}_L F = \overline{\bigsqcap}_{(L^{\mathrm{op}})} F$ and $\overline{\bigsqcap}_L F = \underline{\bigsqcup}_{(L^{\mathrm{op}})} F$.

One can check that every non empty relational structure which is completely-distributive is also complete.

Let us observe that there exists a non empty poset which is completely-distributive, trivial, and strict.

The following proposition is true

(51) For every non empty poset L holds L is completely-distributive iff L^{op} is completely-distributive.

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Irreducible and Prime Elements[1]

Beata Madras

Summary. In the paper open and order generating subsets are defined. Irreducible and prime elements are also defined. The article includes definitions and facts presented in [16, pp. 68–72].

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The articles [29], [25], [1], [15], [28], [30], [31], [9], [23], [2], [24], [4], [11], [12], [10], [13], [3], [27], [21], [22], [5], [18], [6], [14], [33], [19], [20], [8], [17], [32], [26], and [7] provide the notation and terminology for this paper.

1. Preliminaries

In this paper L denotes a lattice and l denotes an element of L.

The scheme *NonUniqExD1* concerns a non empty relational structure $\mathcal{A}$, a subset $\mathcal{B}$ of $\mathcal{A}$, a non empty subset $\mathcal{C}$ of $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

> There exists a function f from $\mathcal{B}$ into $\mathcal{C}$ such that for every element e of $\mathcal{A}$ if $e \in \mathcal{B}$, then there exists an element u of $\mathcal{A}$ such that $u \in \mathcal{C}$ and $u = f(e)$ and $\mathcal{P}[e, u]$

provided the following requirement is met:

- For every element e of $\mathcal{A}$ such that $e \in \mathcal{B}$ there exists an element u of $\mathcal{A}$ such that $u \in \mathcal{C}$ and $\mathcal{P}[e, u]$.

Let L be a lattice, let A be a non empty subset of the carrier of L, let f be a function from A into A, and let n be an element of $\mathbb{N}$. Then f^n is a function from A into A.

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Let L be a lattice, let C, D be non empty subsets of the carrier of L, let f be a function from C into D, and let c be an element of C. Then $f(c)$ is an element of L.

Let L be a non empty poset. One can check that every chain of L is filtered and directed.

Let us observe that there exists a lattice which is strict, continuous, distributive, and lower-bounded.

Next we state three propositions:

(1) Let S, T be semilattices and f be a map from S into T. Then f is meet-preserving if and only if for all elements x, y of S holds $f(x \sqcap y) = f(x) \sqcap f(y)$.

(2) Let S, T be sup-semilattices and f be a map from S into T. Then f is join-preserving if and only if for all elements x, y of S holds $f(x \sqcup y) = f(x) \sqcup f(y)$.

(3) Let S, T be lattices and f be a map from S into T. Suppose T is distributive and f is meet-preserving, join-preserving, and one-to-one. Then S is distributive.

Let S, T be complete lattices. Observe that there exists a map from S into T which is sups-preserving.

The following proposition is true

(4) Let S, T be complete lattices and f be a sups-preserving map from S into T. Suppose T is meet-continuous and f is meet-preserving and one-to-one. Then S is meet-continuous.

2. Open sets

Let L be a non empty reflexive relational structure and let X be a subset of L. We say that X is open if and only if:

(Def. 1) For every element x of L such that $x \in X$ there exists an element y of L such that $y \in X$ and $y \ll x$.

The following two propositions are true:

(5) Let L be an up-complete lattice and X be an upper subset of L. Then X is open if and only if for every element x of L such that $x \in X$ holds $\twoheaddownarrow x \cap X \neq \emptyset$.

(6) Let L be an up-complete lattice and X be an upper subset of L. Then X is open if and only if $X = \bigcup\{\twoheaduparrow x, x$ ranges over elements of L: $x \in X\}$.

Let L be an up-complete lower-bounded lattice. Note that there exists a filter of L which is open.

The following three propositions are true:

(7) For every lower-bounded continuous lattice L and for every element x of L holds $\twoheaduparrow x$ is open.

(8) Let L be a lower-bounded continuous lattice and x, y be elements of L. If $x \ll y$, then there exists an open filter F of L such that $y \in F$ and $F \subseteq \uparrow x$.

(9) Let L be a complete lattice, X be an open upper subset of L, and x be an element of L. If $x \in -X$, then there exists an element m of L such that $x \leqslant m$ and m is maximal in $-X$.

3. Irreducible elements

Let G be a non empty relational structure and let g be an element of G. We say that g is meet-irreducible if and only if:

(Def. 2) For all elements x, y of G such that $g = x \sqcap y$ holds $x = g$ or $y = g$.

We introduce g is irreducible as a synonym of g is meet-irreducible.

Let G be a non empty relational structure and let g be an element of G. We say that g is join-irreducible if and only if:

(Def. 3) For all elements x, y of G such that $g = x \sqcup y$ holds $x = g$ or $y = g$.

Let L be a non empty relational structure. The functor $\mathrm{IRR}(L)$ yielding a subset of L is defined as follows:

(Def. 4) For every element x of L holds $x \in \mathrm{IRR}(L)$ iff x is irreducible.

The following proposition is true

(10) For every upper-bounded antisymmetric non empty relational structure L with g.l.b.'s holds $\top_L$ is irreducible.

Let L be an upper-bounded antisymmetric non empty relational structure with g.l.b.'s. Observe that there exists an element of L which is irreducible.

We now state four propositions:

(11) Let L be a semilattice and x be an element of L. Then x is irreducible if and only if for every finite non empty subset A of L such that $x = \inf A$ holds $x \in A$.

(12) For every lattice L and for every element l of L such that $\uparrow l \setminus \{l\}$ is a filter of L holds l is irreducible.

(13) Let L be a lattice, p be an element of L, and F be a filter of L. If p is maximal in $-F$, then p is irreducible.

(14) Let L be a lower-bounded continuous lattice and x, y be elements of L. Suppose $y \nleqslant x$. Then there exists an element p of L such that p is irreducible and $x \leqslant p$ and $y \nleqslant p$.

4. Order generating sets

Let L be a non empty relational structure and let X be a subset of L. We say that X is order-generating if and only if:

(Def. 5) For every element x of L holds $\inf \uparrow x \cap X$ exists in L and $x = \inf(\uparrow x \cap X)$.

The following propositions are true:

(15) Let L be an up-complete lower-bounded lattice and X be a subset of L. Then X is order-generating if and only if for every element l of L there exists a subset Y of X such that $l = \bigsqcap_L Y$.

(16) Let L be an up-complete lower-bounded lattice and X be a subset of L. Then X is order-generating if and only if for every subset Y of L such that $X \subseteq Y$ and for every subset Z of Y holds $\bigsqcap_L Z \in Y$ holds the carrier of $L = Y$.

(17) Let L be an up-complete lower-bounded lattice and X be a subset of L. Then X is order-generating if and only if for all elements l_1, l_2 of L such that $l_2 \nleqslant l_1$ there exists an element p of L such that $p \in X$ and $l_1 \leqslant p$ and $l_2 \nleqslant p$.

(18) Let L be a lower-bounded continuous lattice and X be a subset of L. If $X = \mathrm{IRR}(L) \setminus \{\top_L\}$, then X is order-generating.

(19) Let L be a lower-bounded continuous lattice and X, Y be subsets of L. If X is order-generating and $X \subseteq Y$, then Y is order-generating.

5. Prime elements

Let L be a non empty relational structure and let l be an element of L. We say that l is prime if and only if:

(Def. 6) For all elements x, y of L such that $x \sqcap y \leqslant l$ holds $x \leqslant l$ or $y \leqslant l$.

Let L be a non empty relational structure. The functor $\mathrm{PRIME}(L)$ yielding a subset of L is defined by:

(Def. 7) For every element x of L holds $x \in \mathrm{PRIME}(L)$ iff x is prime.

Let L be a non empty relational structure and let l be an element of L. We say that l is co-prime if and only if:

(Def. 8) $l^{\smile}$ is prime.

We now state two propositions:

(20) For every upper-bounded antisymmetric non empty relational structure L holds $\top_L$ is prime.

(21) For every lower-bounded antisymmetric non empty relational structure L holds $\bot_L$ is co-prime.

Let L be an upper-bounded antisymmetric non empty relational structure. Note that there exists an element of L which is prime.

The following propositions are true:

(22) Let L be a semilattice and l be an element of L. Then l is prime if and only if for every finite non empty subset A of L such that $l \geqslant \inf A$ there exists an element a of L such that $a \in A$ and $l \geqslant a$.

(23) Let L be a sup-semilattice and x be an element of L. Then x is co-prime if and only if for every finite non empty subset A of L such that $x \leqslant \sup A$ there exists an element a of L such that $a \in A$ and $x \leqslant a$.

(24) For every lattice L and for every element l of L such that l is prime holds l is irreducible.

(25) Let given l. Then l is prime if and only if for arbitrary x and for every map f from L into $2^{\{x\}}_{\subseteq}$ such that for every element p of L holds $f(p) = \emptyset$ iff $p \leqslant l$ holds f is meet-preserving and join-preserving.

(26) Let L be an upper-bounded lattice and l be an element of L. If $l \neq \top_L$, then l is prime iff $-{\downarrow}l$ is a filter of L.

(27) For every distributive lattice L and for every element l of L holds l is prime iff l is irreducible.

(28) For every distributive lattice L holds $\mathrm{PRIME}(L) = \mathrm{IRR}(L)$.

(29) Let L be a Boolean lattice and l be an element of L. Suppose $l \neq \top_L$. Then l is prime if and only if for every element x of L such that $x > l$ holds $x = \top_L$.

(30) Let L be a continuous distributive lower-bounded lattice and l be an element of L. Suppose $l \neq \top_L$. Then l is prime if and only if there exists an open filter F of L such that l is maximal in $-F$.

(31) Let L be a relational structure and X be a subset of the carrier of L. Then $\chi_{X,\text{the carrier of } L}$ is a map from L into $2^{\{\emptyset\}}_{\subseteq}$.

(32) Let L be a non empty relational structure and p, x be elements of L. Then $\chi_{-\downarrow p,\text{the carrier of } L}(x) = \emptyset$ if and only if $x \leqslant p$.

(33) Let L be an upper-bounded lattice, f be a map from L into $2^{\{\emptyset\}}_{\subseteq}$, and p be a prime element of L. Suppose $\chi_{-\downarrow p,\text{the carrier of } L} = f$. Then f is meet-preserving and join-preserving.

(34) For every complete lattice L such that $\mathrm{PRIME}(L)$ is order-generating holds L is distributive and meet-continuous.

(35) For every lower-bounded continuous lattice L holds L is distributive iff $\mathrm{PRIME}(L)$ is order-generating.

(36) For every lower-bounded continuous lattice L holds L is distributive iff L is Heyting.

(37) Let L be a continuous complete lattice. Suppose that for every element l of L there exists a subset X of L such that $l = \sup X$ and for every element x of L such that $x \in X$ holds x is co-prime. Let l be an element of L. Then $l = \bigsqcup_L({\downarrow}l \cap \mathrm{PRIME}(L^{\mathrm{op}}))$.

(38) Let L be a complete lattice. Then L is completely-distributive if and only if the following conditions are satisfied:

(i) L is continuous, and

(ii) for every element l of L there exists a subset X of L such that $l = \sup X$ and for every element x of L such that $x \in X$ holds x is co-prime.

(39) Let L be a complete lattice. Then L is completely-distributive if and only if L is distributive and continuous and L^{op} is continuous.

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Prime Ideals and Filters[1]

Grzegorz Bancerek
Warsaw University
Białystok

Summary. The part of [12, pp. 73–77], i.e. definitions and propositions 3.16–3.27, is formalized in the paper.

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The notation and terminology used in this paper are introduced in the following articles: [22], [25], [8], [24], [19], [26], [27], [7], [11], [6], [20], [10], [15], [21], [23], [1], [2], [3], [14], [9], [16], [17], [5], [4], [18], and [13].

1. The lattice of subsets

One can prove the following propositions:

(1) For every complete lattice L and for every ideal I of L holds $\bot_L \in I$.

(2) For every upper-bounded non empty poset L and for every filter F of L holds $\top_L \in F$.

(3) For every complete lattice L and for all sets X, Y such that $X \subseteq Y$ holds $\bigsqcup_L X \leqslant \bigsqcup_L Y$ and $\bigsqcap_L X \geqslant \bigsqcap_L Y$.

(4) For every set X holds the carrier of $2^X_\subseteq = 2^X$.

(5) For every bounded antisymmetric non empty relational structure L holds L is trivial iff $\top_L = \bot_L$.

Let X be a set. Note that $2^X_\subseteq$ is Boolean.

Let X be a non empty set. Note that $2^X_\subseteq$ is non trivial.

We now state three propositions:

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(6) For every upper-bounded non empty poset L holds $\{\top_L\} = \uparrow(\top_L)$.

(7) For every lower-bounded non empty poset L holds $\{\bot_L\} = \downarrow(\bot_L)$.

(8) For every lower-bounded non empty poset L and for every filter F of L holds F is proper iff $\bot_L \notin F$.

One can verify that there exists a lattice which is non trivial, Boolean, and strict.

Let L be a non trivial upper-bounded non empty poset. One can check that there exists a filter of L which is proper.

Next we state several propositions:

(9) For every set X and for every element a of $2^X_\subseteq$ holds $\neg a = X \setminus a$.

(10) Let X be a set and Y be a subset of $2^X_\subseteq$. Then Y is lower if and only if for all sets x, y such that $x \subseteq y$ and $y \in Y$ holds $x \in Y$.

(11) Let X be a set and Y be a subset of $2^X_\subseteq$. Then Y is upper if and only if for all sets x, y such that $x \subseteq y$ and $y \subseteq X$ and $x \in Y$ holds $y \in Y$.

(12) Let X be a set and Y be a lower subset of $2^X_\subseteq$. Then Y is directed if and only if for all sets x, y such that $x \in Y$ and $y \in Y$ holds $x \cup y \in Y$.

(13) Let X be a set and Y be an upper subset of $2^X_\subseteq$. Then Y is filtered if and only if for all sets x, y such that $x \in Y$ and $y \in Y$ holds $x \cap y \in Y$.

(14) Let X be a set and Y be a non empty lower subset of $2^X_\subseteq$. Then Y is directed if and only if for every finite family Z of subsets of X such that $Z \subseteq Y$ holds $\bigcup Z \in Y$.

(15) Let X be a set and Y be a non empty upper subset of $2^X_\subseteq$. Then Y is filtered if and only if for every finite family Z of subsets of X such that $Z \subseteq Y$ holds $\mathrm{Intersect}(Z) \in Y$.

2. Prime ideals and filters

Let L be a poset with g.l.b.'s and let I be an ideal of L. We say that I is prime if and only if:

(Def. 1) For all elements x, y of L such that $x \sqcap y \in I$ holds $x \in I$ or $y \in I$.

One can prove the following proposition

(16) Let L be a poset with g.l.b.'s and I be an ideal of L. Then I is prime if and only if for every finite non empty subset A of L such that $\inf A \in I$ there exists an element a of L such that $a \in A$ and $a \in I$.

Let L be a lattice. Note that there exists an ideal of L which is prime.

Next we state the proposition

(17) Let L_1, L_2 be lattices. Suppose the relational structure of L_1 = the relational structure of L_2. Let x be a set. If x is a prime ideal of L_1, then x is a prime ideal of L_2.

Let L be a poset with l.u.b.'s and let F be a filter of L. We say that F is prime if and only if:

(Def. 2) For all elements x, y of L such that $x \sqcup y \in F$ holds $x \in F$ or $y \in F$.

Next we state the proposition

(18) Let L be a poset with l.u.b.'s and F be a filter of L. Then F is prime if and only if for every finite non empty subset A of L such that $\sup A \in F$ there exists an element a of L such that $a \in A$ and $a \in F$.

Let L be a lattice. One can verify that there exists a filter of L which is prime.

The following propositions are true:

(19) Let L_1, L_2 be lattices. Suppose the relational structure of L_1 = the relational structure of L_2. Let x be a set. If x is a prime filter of L_1, then x is a prime filter of L_2.

(20) Let L be a lattice and x be a set. Then x is a prime ideal of L if and only if x is a prime filter of $L^{\rm op}$.

(21) Let L be a lattice and x be a set. Then x is a prime filter of L if and only if x is a prime ideal of $L^{\rm op}$.

(22) Let L be a poset with g.l.b.'s and I be an ideal of L. Then I is prime if and only if one of the following conditions is satisfied:

(i) $-I$ is a filter of L, or

(ii) $-I = \emptyset$.

(23) For every lattice L and for every ideal I of L holds I is prime iff $I \in \mathrm{PRIME}(\langle \mathrm{Ids}(L), \subseteq \rangle)$.

(24) Let L be a Boolean lattice and F be a filter of L. Then F is prime if and only if for every element a of L holds $a \in F$ or $\neg a \in F$.

(25) Let X be a set and F be a filter of $2^X_\subseteq$. Then F is prime if and only if for every subset A of X holds $A \in F$ or $X \setminus A \in F$.

Let L be a non empty poset and let F be a filter of L. We say that F is ultra if and only if:

(Def. 3) F is proper and for every filter G of L such that $F \subseteq G$ holds $F = G$ or G = the carrier of L.

Let L be a non empty poset. Note that every filter of L which is ultra is also proper.

The following propositions are true:

(26) For every Boolean lattice L and for every filter F of L holds F is proper and prime iff F is ultra.

(27) Let L be a distributive lattice, I be an ideal of L, and F be a filter of L. Suppose I misses F. Then there exists an ideal P of L such that P is prime and $I \subseteq P$ and P misses F.

(28) Let L be a distributive lattice, I be an ideal of L, and x be an element of L. If $x \notin I$, then there exists an ideal P of L such that P is prime and $I \subseteq P$ and $x \notin P$.

(29) Let L be a distributive lattice, I be an ideal of L, and F be a filter of L. Suppose I misses F. Then there exists a filter P of L such that P is

prime and $F \subseteq P$ and I misses P.

(30) Let L be a non trivial Boolean lattice and F be a proper filter of L. Then there exists a filter G of L such that $F \subseteq G$ and G is ultra.

3. Cluster points of a filter of sets

Let T be a topological space and let F, x be sets. We say that x is a cluster point of F, T if and only if:

(Def. 4) For every subset A of T such that A is open and $x \in A$ and for every set B such that $B \in F$ holds A meets B.

We say that x is a convergence point of F, T if and only if:

(Def. 5) For every subset A of T such that A is open and $x \in A$ holds $A \in F$.

Let X be a non empty set. Note that there exists a filter of $2^X_\subseteq$ which is ultra.

We now state several propositions:

(31) Let T be a non empty topological space, F be an ultra filter of $2^{\text{the carrier of } T}_\subseteq$, and p be a set. Then p is a cluster point of F, T if and only if p is a convergence point of F, T.

(32) Let T be a non empty topological space and x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose $x \ll y$. Let F be a proper filter of $2^{\text{the carrier of } T}_\subseteq$. Suppose $x \in F$. Then there exists an element p of T such that $p \in y$ and p is a cluster point of F, T.

(33) Let T be a non empty topological space and x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose $x \ll y$. Let F be an ultra filter of $2^{\text{the carrier of } T}_\subseteq$. Suppose $x \in F$. Then there exists an element p of T such that $p \in y$ and p is a convergence point of F, T.

(34) Let T be a non empty topological space and x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose that

(i) $x \subseteq y$, and

(ii) for every ultra filter F of $2^{\text{the carrier of } T}_\subseteq$ such that $x \in F$ there exists an element p of T such that $p \in y$ and p is a convergence point of F, T.

Then $x \ll y$.

(35) Let T be a non empty topological space, B be a prebasis of T, and x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose $x \subseteq y$. Then $x \ll y$ if and only if for every subset F of B such that $y \subseteq \bigcup F$ there exists a finite subset G of F such that $x \subseteq \bigcup G$.

(36) Let L be a distributive complete lattice and x, y be elements of L. Then $x \ll y$ if and only if for every prime ideal P of L such that $y \leqslant \sup P$ holds $x \in P$.

(37) For every lattice L and for every element p of L such that p is prime holds $\downarrow p$ is prime.

4. Pseudo prime elements

Let L be a lattice and let p be an element of L. We say that p is pseudoprime if and only if:

(Def. 6) There exists a prime ideal P of L such that $p = \sup P$.

We now state several propositions:

(38) For every lattice L and for every element p of L such that p is prime holds p is pseudoprime.

(39) Let L be a continuous lattice and p be an element of L. Suppose p is pseudoprime. Let A be a finite non empty subset of L. If $\inf A \ll p$, then there exists an element a of L such that $a \in A$ and $a \leqslant p$.

(40) Let L be a continuous lattice and p be an element of L. Suppose that
(i) $p \neq \top_L$ or $\top_L$ is not compact, and
(ii) for every finite non empty subset A of L such that $\inf A \ll p$ there exists an element a of L such that $a \in A$ and $a \leqslant p$.
Then $\uparrow$fininfs$(-\downarrow p)$ misses $\downarrow p$.

(41) Let L be a continuous lattice. Suppose $\top_L$ is compact. Then
(i) for every finite non empty subset A of L such that $\inf A \ll \top_L$ there exists an element a of L such that $a \in A$ and $a \leqslant \top_L$, and
(ii) $\uparrow$fininfs$(-\downarrow(\top_L))$ meets $\downarrow(\top_L)$.

(42) Let L be a continuous lattice and p be an element of L. Suppose $\uparrow$fininfs$(-\downarrow p)$ misses $\downarrow p$. Let A be a finite non empty subset of L. If $\inf A \ll p$, then there exists an element a of L such that $a \in A$ and $a \leqslant p$.

(43) Let L be a distributive continuous lattice and p be an element of L. If $\uparrow$fininfs$(-\downarrow p)$ misses $\downarrow p$, then p is pseudoprime.

Let L be a non empty relational structure and let R be a binary relation on the carrier of L. We say that R is multiplicative if and only if:

(Def. 7) For all elements a, x, y of L such that $\langle a, x\rangle \in R$ and $\langle a, y\rangle \in R$ holds $\langle a, x \sqcap y\rangle \in R$.

Let L be a lower-bounded sup-semilattice, let R be an auxiliary binary relation on L, and let x be an element of L. Observe that $\uparrow_R x$ is upper.

We now state several propositions:

(44) Let L be a lower-bounded lattice and R be an auxiliary binary relation on L. Then R is multiplicative if and only if for every element x of L holds $\uparrow_R x$ is filtered.

(45) Let L be a lower-bounded lattice and R be an auxiliary binary relation on L. Then R is multiplicative if and only if for all elements a, b, x, y of L such that $\langle a, x\rangle \in R$ and $\langle b, y\rangle \in R$ holds $\langle a \sqcap b, x \sqcap y\rangle \in R$.

(46) Let L be a lower-bounded lattice and R be an auxiliary binary relation on L. Then R is multiplicative if and only if for every full relational

substructure S of $[: L, L :]$ such that the carrier of $S = R$ holds S is meet-inheriting.

(47) Let L be a lower-bounded lattice and R be an auxiliary binary relation on L. Then R is multiplicative if and only if $\downarrow R$ is meet-preserving.

(48) Let L be a continuous lower-bounded lattice. Suppose $\ll_L$ is multiplicative. Let p be an element of L. Then p is pseudoprime if and only if for all elements a, b of L such that $a \sqcap b \ll p$ holds $a \leqslant p$ or $b \leqslant p$.

(49) Let L be a continuous lower-bounded lattice. Suppose $\ll_L$ is multiplicative. Let p be an element of L. If p is pseudoprime, then p is prime.

(50) Let L be a distributive continuous lower-bounded lattice. Suppose that for every element p of L such that p is pseudoprime holds p is prime. Then $\ll_L$ is multiplicative.

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Algebraic Lattices

Robert Milewski
Warsaw University
Białystok

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The articles [18], [20], [16], [10], [21], [7], [19], [13], [8], [1], [17], [2], [3], [12], [22], [4], [9], [6], [11], [14], [5], and [15] provide the notation and terminology for this paper.

1. The Subset of All Compact Elements

Let L be a non empty reflexive relational structure. The functor CompactSublatt(L) yields a strict full relational substructure of L and is defined as follows:

(Def. 1) For every element x of L holds $x \in$ the carrier of CompactSublatt(L) iff x is compact.

Let L be a lower-bounded non empty reflexive antisymmetric relational structure. Observe that CompactSublatt(L) is non empty.

Next we state three propositions:

(1) Let L be a complete lattice and x, y, k be elements of L. If $x \leqslant k$ and $k \leqslant y$ and $k \in$ the carrier of CompactSublatt(L), then $x \ll y$.

(2) Let L be a complete lattice and x be an element of L. Then $\uparrow x$ is an open filter of L if and only if x is compact.

(3) For every lower-bounded non empty poset L with l.u.b.'s holds CompactSublatt(L) is join-inheriting and $\bot_L \in$ the carrier of CompactSublatt(L).

Let L be a non empty reflexive relational structure and let x be an element of L. The functor compactbelow(x) yielding a subset of L is defined by:

(Def. 2) compactbelow(x) $= \{y, y$ ranges over elements of L: $x \geqslant y \ \wedge \ y$ is compact$\}$.

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We now state three propositions:

(4) Let L be a non empty reflexive relational structure and x, y be elements of L. Then $y \in \mathrm{compactbelow}(x)$ if and only if the following conditions are satisfied:

(i) $x \geqslant y$, and

(ii) y is compact.

(5) For every non empty reflexive relational structure L and for every element x of L holds $\mathrm{compactbelow}(x) = {\downarrow}x \cap$ the carrier of $\mathrm{CompactSublatt}(L)$.

(6) For every non empty reflexive transitive relational structure L and for every element x of L holds $\mathrm{compactbelow}(x) \subseteq {\downarrow}x$.

Let L be a non empty lower-bounded reflexive antisymmetric relational structure and let x be an element of L. Note that $\mathrm{compactbelow}(x)$ is non empty.

2. Algebraic Lattices

Let L be a non empty reflexive relational structure. We say that L satisfies axiom K if and only if:

(Def. 3) For every element x of L holds $x = \sup \mathrm{compactbelow}(x)$.

Let L be a non empty reflexive relational structure. We say that L is algebraic if and only if:

(Def. 4) For every element x of L holds $\mathrm{compactbelow}(x)$ is non empty and directed and L is up-complete and satisfies axiom K.

We now state the proposition

(7) Let L be a lattice. Then L is algebraic if and only if the following conditions are satisfied:

(i) L is continuous, and

(ii) for all elements x, y of L such that $x \ll y$ there exists an element k of L such that $k \in$ the carrier of $\mathrm{CompactSublatt}(L)$ and $x \leqslant k$ and $k \leqslant y$.

Let us observe that every lattice which is algebraic is also continuous.

Let us note that every non empty reflexive relational structure which is algebraic is also up-complete and satisfies axiom K.

Let L be a non empty poset with l.u.b.'s. One can check that $\mathrm{CompactSublatt}(L)$ is join-inheriting.

Let L be a lattice. We say that L is arithmetic if and only if:

(Def. 5) L is algebraic and $\mathrm{CompactSublatt}(L)$ is meet-inheriting.

3. Arithmetic Lattices

Let us note that every lattice which is arithmetic is also algebraic.

Let us note that every lattice which is trivial is also arithmetic.

Let us note that there exists a lattice which is trivial and strict.

We now state a number of propositions:

(8) Let L_1, L_2 be non empty reflexive antisymmetric relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is up-complete. Let x_1, y_1 be elements of L_1 and x_2, y_2 be elements of L_2. If $x_1 = x_2$ and $y_1 = y_2$ and $x_1 \ll y_1$, then $x_2 \ll y_2$.

(9) Let L_1, L_2 be non empty reflexive antisymmetric relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is up-complete. Let x be an element of L_1 and y be an element of L_2. If $x = y$ and x is compact, then y is compact.

(10) Let L_1, L_2 be up-complete non empty reflexive antisymmetric relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2. Let x be an element of L_1 and y be an element of L_2. If $x = y$, then $\text{compactbelow}(x) = \text{compactbelow}(y)$.

(11) Let L_1, L_2 be relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is non empty. Then L_2 is non empty.

(12) Let L_1, L_2 be non empty relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is reflexive. Then L_2 is reflexive.

(13) Let L_1, L_2 be relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is transitive. Then L_2 is transitive.

(14) Let L_1, L_2 be relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is antisymmetric. Then L_2 is antisymmetric.

(15) Let L_1, L_2 be non empty reflexive relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is up-complete. Then L_2 is up-complete.

(16) For all up-complete non empty reflexive antisymmetric relational structures L_1, L_2 such that the relational structure of $L_1 =$ the relational structure of L_2 and L_1 satisfies axiom K and for every element x of L_1 holds $\text{compactbelow}(x)$ is non empty and directed holds L_2 satisfies axiom K.

(17) Let L_1, L_2 be non empty reflexive antisymmetric relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is algebraic. Then L_2 is algebraic.

(18) Let L_1, L_2 be lattices. Suppose the relational structure of L_1 = the relational structure of L_2 and L_1 is arithmetic. Then L_2 is arithmetic.

Let L be a non empty relational structure. Observe that the relational structure of L is non empty.

Let L be a non empty reflexive relational structure. One can check that the relational structure of L is reflexive.

Let L be a transitive relational structure. Note that the relational structure of L is transitive.

Let L be an antisymmetric relational structure. Observe that the relational structure of L is antisymmetric.

Let L be a relational structure with g.l.b.'s. Note that the relational structure of L has g.l.b.'s.

Let L be a relational structure with l.u.b.'s. One can check that the relational structure of L has l.u.b.'s.

Let L be an up-complete non empty reflexive relational structure. One can check that the relational structure of L is up-complete.

Let L be an algebraic non empty reflexive antisymmetric relational structure. Note that the relational structure of L is algebraic.

Let L be an arithmetic lattice. One can verify that the relational structure of L is arithmetic.

Next we state several propositions:

(19) Let L be a non empty transitive relational structure, S be a non empty full relational substructure of L, and X be a subset of S. Suppose sup X exists in L and $\bigsqcup_L X$ is an element of S. Then sup X exists in S and $\sup X = \bigsqcup_L X$.

(20) Let L be a non empty transitive relational structure, S be a non empty full relational substructure of L, and X be a subset of S. Suppose inf X exists in L and $\bigsqcap_L X$ is an element of S. Then inf X exists in S and $\inf X = \bigsqcap_L X$.

(21) For every algebraic lattice L holds L is arithmetic iff $\mathrm{CompactSublatt}(L)$ is a lattice.

(22) For every algebraic lower-bounded lattice L holds L is arithmetic iff $\ll_L$ is multiplicative.

(23) Let L be an arithmetic lower-bounded lattice and p be an element of L. If p is pseudoprime, then p is prime.

(24) Let L be an algebraic distributive lower-bounded lattice. Suppose that for every element p of L such that p is pseudoprime holds p is prime. Then L is arithmetic.

Let L be an algebraic lattice and let c be a closure map from L into L. Note that there exists a subset of $\mathrm{Im}\, c$ which is non empty and directed.

We now state three propositions:

(25) Let L be an algebraic lattice and c be a closure map from L into L. If c is directed-sups-preserving, then $c^\circ(\Omega_{\mathrm{CompactSublatt}(L)}) \subseteq \Omega_{\mathrm{CompactSublatt}(\mathrm{Im}\, c)}$.

(26) Let L be an algebraic lower-bounded lattice and c be a closure map from L into L. If c is directed-sups-preserving, then $\mathrm{Im}\, c$ is an algebraic lattice.

(27) Let L be an algebraic lower-bounded lattice and c be a closure map from L into L. If c is directed-sups-preserving, then $c^{\circ}(\Omega_{\mathrm{CompactSublatt}(L)}) = \Omega_{\mathrm{CompactSublatt}(\mathrm{Im}\, c)}$.

4. Boolean Posets as Algebraic Lattices

Next we state several propositions:

(28) For all sets X, x holds x is an element of $2^X_\subseteq$ iff $x \subseteq X$.

(29) Let X be a set and x, y be elements of $2^X_\subseteq$. Then $x \ll y$ if and only if for every family Y of subsets of X such that $y \subseteq \bigcup Y$ there exists a finite subset Z of Y such that $x \subseteq \bigcup Z$.

(30) For every set X and for every element x of $2^X_\subseteq$ holds x is finite iff x is compact.

(31) For every set X and for every element x of $2^X_\subseteq$ holds $\mathrm{compactbelow}(x) = \{y : y$ ranges over finite subsets of $x\}$.

(32) For every set X and for every subset F of X holds $F \in$ the carrier of $\mathrm{CompactSublatt}(2^X_\subseteq)$ iff F is finite.

Let X be a set and let x be an element of $2^X_\subseteq$. Observe that $\mathrm{compactbelow}(x)$ is lower and directed.

The following proposition is true

(33) For every set X holds $2^X_\subseteq$ is algebraic.

Let X be a set. Observe that $2^X_\subseteq$ is algebraic.

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Reconstructions of Special Sequences[1]

Yatsuka Nakamura
Shinshu University
Nagano

Roman Matuszewski
Warsaw University
Białystok

Summary. We discuss here some methods for reconstructing special sequences which generate special polygonal arcs in $\mathcal{E}^2_{\mathrm{T}}$. For such reconstructions we introduce a "mid" function which cuts out the middle part of a sequence; the "$\downharpoonright$" function, which cuts down the left part of a sequence at some point; the "$\downharpoonleft$" function for cutting down the right part at some point; and the "$\downharpoonright\downharpoonleft$" function for cutting down both sides at two given points.

We also introduce some methods glueing two special sequences. By such cutting and glueing methods, the speciality of sequences (generatability of special polygonal arcs) is shown to be preserved.

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The papers [12], [15], [14], [8], [1], [9], [10], [2], [3], [13], [4], [16], [7], [11], [6], and [5] provide the notation and terminology for this paper.

1. Preliminaries

We adopt the following convention: n, i, i_1, i_2, j denote natural numbers and D denotes a non empty set.

We now state a number of propositions:

(1) For all natural numbers i, i_1 such that $i \geqslant i_1$ or $i -' i_1 \geqslant 1$ or $i - i_1 \geqslant 1$ holds $i -' i_1 = i - i_1$.

(2) For every natural number n holds $n -' 0 = n$.

(3) For all natural numbers i_1, i_2 holds $i_1 - i_2 \leqslant i_1 -' i_2$.

(4) For all natural numbers n, i_1, i_2 such that $i_1 \leqslant i_2$ holds $n -' i_2 \leqslant n -' i_1$.

[1]The work has been done while the second author was visiting Nagano in autumn 1996.

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(5) For all n, i_1, i_2 such that $i_1 \leqslant i_2$ holds $i_1 -' n \leqslant i_2 -' n$.

(6) For all natural numbers i, i_1 such that $i \geqslant i_1$ or $i -' i_1 \geqslant 1$ or $i - i_1 \geqslant 1$ holds $i -' i_1 + i_1 = i$.

(7) For all natural numbers i_1, i_2 such that $i_1 \leqslant i_2$ holds $i_1 -' 1 \leqslant i_2$.

(8) For every i holds $i -' 2 = i -' 1 -' 1$.

(9) For all i_1, i_2 such that $i_1 + 1 \leqslant i_2$ holds $i_1 < i_2$ and $i_1 -' 1 < i_2$ and $i_1 -' 2 < i_2$ and $i_1 \leqslant i_2$.

(10) Let given i_1, i_2. Suppose $i_1 + 2 \leqslant i_2$ or $i_1 + 1 + 1 \leqslant i_2$. Then $i_1 + 1 < i_2$ and $(i_1+1) -' 1 < i_2$ and $(i_1+1) -' 2 < i_2$ and $i_1 + 1 \leqslant i_2$ and $i_1 -' 1 + 1 < i_2$ and $(i_1 -' 1 + 1) -' 1 < i_2$ and $i_1 < i_2$ and $i_1 -' 1 < i_2$ and $i_1 -' 2 < i_2$ and $i_1 \leqslant i_2$.

(11) For all i_1, i_2 such that $i_1 \leqslant i_2$ or $i_1 \leqslant i_2 -' 1$ holds $i_1 < i_2 + 1$ and $i_1 \leqslant i_2 + 1$ and $i_1 < i_2 + 1 + 1$ and $i_1 \leqslant i_2 + 1 + 1$ and $i_1 < i_2 + 2$ and $i_1 \leqslant i_2 + 2$.

(12) For all i_1, i_2 such that $i_1 < i_2$ or $i_1 + 1 \leqslant i_2$ holds $i_1 \leqslant i_2 -' 1$.

(13) For all i, i_1, i_2 such that $i \geqslant i_1$ holds $i \geqslant i_1 -' i_2$.

(14) For all i, i_1 such that $1 \leqslant i$ and $1 \leqslant i_1 -' i$ holds $i_1 -' i < i_1$.

(15) For all finite sequences p, q and for every i such that $\operatorname{len} p < i$ but $i \leqslant \operatorname{len} p + \operatorname{len} q$ or $i \leqslant \operatorname{len}(p \frown q)$ holds $(p \frown q)(i) = q(i - \operatorname{len} p)$.

(16) Let x be arbitrary and f be a finite sequence of elements of D. Then $\operatorname{len}(f \frown \langle x \rangle) = \operatorname{len} f + 1$ and $\operatorname{len}(\langle x \rangle \frown f) = \operatorname{len} f + 1$ and $(f \frown \langle x \rangle)(\operatorname{len} f + 1) = x$ and $(\langle x \rangle \frown f)(1) = x$.

(17) Let x be arbitrary and f be a finite sequence of elements of D. Suppose $1 \leqslant \operatorname{len} f$. Then $(f \frown \langle x \rangle)(1) = f(1)$ and $(f \frown \langle x \rangle)(1) = \pi_1 f$ and $(\langle x \rangle \frown f)(\operatorname{len} f + 1) = f(\operatorname{len} f)$ and $(\langle x \rangle \frown f)(\operatorname{len} f + 1) = \pi_{\operatorname{len} f} f$.

(18) For every finite sequence f of elements of D such that $\operatorname{len} f = 1$ holds $\operatorname{Rev}(f) = f$.

(19) For every finite sequence f of elements of D and for every natural number k holds $\operatorname{len}(f_{\lfloor k}) = \operatorname{len} f -' k$.

(20) Let f be a finite sequence of elements of D and k be a natural number. If $1 \leqslant k$ and $k \leqslant n$ and $n \leqslant \operatorname{len} f$, then $(f \restriction n)(k) = f(k)$.

(21) For every finite sequence f of elements of D and for all natural numbers l_1, l_2 holds $f_{\lfloor l_1} \restriction l_2 -' l_1 = (f \restriction l_2)_{\lfloor l_1}$.

2. Middle Function for Finite Sequences

Let us consider D, let f be a finite sequence of elements of D, and let k_1, k_2 be natural numbers. The functor $\operatorname{mid}(f, k_1, k_2)$ yields a finite sequence of elements of D and is defined by:

(Def. 1)(i) $\operatorname{mid}(f, k_1, k_2) = f_{\lfloor k_1 -' 1} \restriction (k_2 -' k_1 + 1)$ if $k_1 \leqslant k_2$,

(ii) $\operatorname{mid}(f, k_1, k_2) = \operatorname{Rev}(f_{\lfloor k_2 -' 1} \restriction (k_1 -' k_2 + 1))$, otherwise.

The following propositions are true:

(22) Let f be a finite sequence of elements of D and k_1, k_2 be natural numbers. If $1 \leqslant k_1$ and $k_1 \leqslant \operatorname{len} f$ and $1 \leqslant k_2$ and $k_2 \leqslant \operatorname{len} f$, then $\operatorname{Rev}(\operatorname{mid}(f, k_1, k_2)) = \operatorname{mid}(\operatorname{Rev}(f), \operatorname{len} f -' k_2 + 1, \operatorname{len} f -' k_1 + 1)$.

(23) Let n, m be natural numbers and f be a finite sequence of elements of D. If $1 \leqslant m$ and $m + n \leqslant \operatorname{len} f$, then $f_{\restriction n}(m) = f(m + n)$ and $f_{\restriction n}(m) = f(n + m)$.

(24) Let i be a natural number and f be a finite sequence of elements of D. If $1 \leqslant i$ and $i \leqslant \operatorname{len} f$, then $(\operatorname{Rev}(f))(i) = f((\operatorname{len} f - i) + 1)$.

(25) For every finite sequence f of elements of D and for every natural number k such that $1 \leqslant k$ holds $\operatorname{mid}(f, 1, k) = f{\upharpoonright}k$.

(26) For every finite sequence f of elements of D and for every natural number k such that $k \leqslant \operatorname{len} f$ holds $\operatorname{mid}(f, k, \operatorname{len} f) = f_{\restriction k -' 1}$.

(27) Let f be a finite sequence of elements of D and k_1, k_2 be natural numbers. Suppose $1 \leqslant k_1$ and $k_1 \leqslant \operatorname{len} f$ and $1 \leqslant k_2$ and $k_2 \leqslant \operatorname{len} f$. Then

(i) $(\operatorname{mid}(f, k_1, k_2))(1) = f(k_1)$,

(ii) if $k_1 \leqslant k_2$, then $\operatorname{len} \operatorname{mid}(f, k_1, k_2) = k_2 -' k_1 + 1$ and for every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{mid}(f, k_1, k_2)$ holds $(\operatorname{mid}(f, k_1, k_2))(i) = f((i + k_1) -' 1)$, and

(iii) if $k_1 > k_2$, then $\operatorname{len} \operatorname{mid}(f, k_1, k_2) = k_1 -' k_2 + 1$ and for every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{mid}(f, k_1, k_2)$ holds $(\operatorname{mid}(f, k_1, k_2))(i) = f(k_1 -' i + 1)$.

(28) For every finite sequence f of elements of D and for all natural numbers k_1, k_2 such that $1 \leqslant \operatorname{len} f$ holds $\operatorname{rng} \operatorname{mid}(f, k_1, k_2) \subseteq \operatorname{rng} f$.

(29) For every finite sequence f of elements of D such that $1 \leqslant \operatorname{len} f$ holds $\operatorname{mid}(f, 1, \operatorname{len} f) = f$.

(30) For every finite sequence f of elements of D such that $1 \leqslant \operatorname{len} f$ holds $\operatorname{mid}(f, \operatorname{len} f, 1) = \operatorname{Rev}(f)$.

(31) Let f be a finite sequence of elements of D and k_1, k_2, i be natural numbers. Suppose $1 \leqslant k_1$ and $k_1 \leqslant k_2$ and $k_2 \leqslant \operatorname{len} f$ and $1 \leqslant i$ and $i \leqslant k_2 -' k_1 + 1$ or $i \leqslant (k_2 - k_1) + 1$ or $i \leqslant (k_2 + 1) - k_1$. Then $(\operatorname{mid}(f, k_1, k_2))(i) = f((i + k_1) -' 1)$ and $(\operatorname{mid}(f, k_1, k_2))(i) = f(i -' 1 + k_1)$ and $(\operatorname{mid}(f, k_1, k_2))(i) = f((i + k_1) - 1)$ and $(\operatorname{mid}(f, k_1, k_2))(i) = f((i - 1) + k_1)$.

(32) Let f be a finite sequence of elements of D and k, i be natural numbers. If $1 \leqslant i$ and $i \leqslant k$ and $k \leqslant \operatorname{len} f$, then $(\operatorname{mid}(f, 1, k))(i) = f(i)$.

(33) Let f be a finite sequence of elements of D and k_1, k_2 be natural numbers. If $1 \leqslant k_1$ and $k_1 \leqslant k_2$ and $k_2 \leqslant \operatorname{len} f$, then $\operatorname{len} \operatorname{mid}(f, k_1, k_2) \leqslant \operatorname{len} f$.

(34) For every finite sequence f of elements of $\mathcal{E}^n_{\mathrm{T}}$ such that $2 \leqslant \operatorname{len} f$ holds $f(1) \in \widetilde{\mathcal{L}}(f)$ and $\pi_1 f \in \widetilde{\mathcal{L}}(f)$ and $f(\operatorname{len} f) \in \widetilde{\mathcal{L}}(f)$ and $\pi_{\operatorname{len} f} f \in \widetilde{\mathcal{L}}(f)$.

(35) For every finite sequence f of elements of $\mathcal{E}^n_{\mathrm{T}}$ and for every natural number i holds $\mathcal{L}(f, i) \subseteq \widetilde{\mathcal{L}}(f)$.

(36) For every finite sequence f of elements of $\mathcal{E}^n_{\mathrm{T}}$ such that $\mathrm{len}\, f \geqslant 2$ holds $f(1) \in \widetilde{\mathcal{L}}(f)$ and $\pi_1 f \in \widetilde{\mathcal{L}}(f)$ and $f(\mathrm{len}\, f) \in \widetilde{\mathcal{L}}(f)$ and $\pi_{\mathrm{len}\, f} f \in \widetilde{\mathcal{L}}(f)$.

(37) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ but $q_1 \in \mathcal{L}(p_1, p_2)$ but $q_2 \in \mathcal{L}(p_1, p_2)$ holds $(q_1)_{\mathbf{1}} = (q_2)_{\mathbf{1}}$ or $(q_1)_{\mathbf{2}} = (q_2)_{\mathbf{2}}$.

(38) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ but $\mathcal{L}(q_1, q_2) \subseteq \mathcal{L}(p_1, p_2)$ holds $(q_1)_{\mathbf{1}} = (q_2)_{\mathbf{1}}$ or $(q_1)_{\mathbf{2}} = (q_2)_{\mathbf{2}}$.

(39) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. If $2 \leqslant n$ and f is a special sequence, then $f{\restriction}n$ is a special sequence.

(40) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Suppose $n \leqslant \mathrm{len}\, f$ and $2 \leqslant \mathrm{len}\, f -' n$ and f is a special sequence. Then $f_{\downharpoonright n}$ is a special sequence.

(41) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and k_1, k_2 be natural numbers. Suppose f is a special sequence and $1 \leqslant k_1$ and $k_1 \leqslant \mathrm{len}\, f$ and $1 \leqslant k_2$ and $k_2 \leqslant \mathrm{len}\, f$ and $k_1 \neq k_2$. Then $\mathrm{mid}(f, k_1, k_2)$ is a special sequence.

3. A Concept of Index for Finite Sequences in $\mathcal{E}^2_{\mathrm{T}}$

Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Let us assume that f is a special sequence and there exists a natural number i such that $1 \leqslant i$ and $i + 1 \leqslant \mathrm{len}\, f$ and $p \in \mathcal{L}(f, i)$. The functor $\mathrm{Index}(p, f)$ yielding a natural number is defined as follows:

(Def. 2) $1 \leqslant \mathrm{Index}(p, f)$ and $\mathrm{Index}(p, f) + 1 \leqslant \mathrm{len}\, f$ and $p \in \mathcal{L}(f, \mathrm{Index}(p, f))$ and $p \neq f(\mathrm{Index}(p, f) + 1)$ or $\mathrm{Index}(p, f) = \mathrm{len}\, f$ and $p = f(\mathrm{len}\, f)$.

One can prove the following propositions:

(42) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$. Then $1 \leqslant \mathrm{Index}(p, f)$ and $\mathrm{Index}(p, f)+1 \leqslant \mathrm{len}\, f$ and $p \in \mathcal{L}(f, \mathrm{Index}(p, f))$ and $p \neq f(\mathrm{Index}(p, f)+1)$ or $\mathrm{Index}(p, f) = \mathrm{len}\, f$ and $p = f(\mathrm{len}\, f)$.

(43) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and there exists a natural number i such that $1 \leqslant i$ and $i + 1 \leqslant \mathrm{len}\, f$ and $p \in \mathcal{L}(f, i)$. Then $1 \leqslant \mathrm{Index}(p, f)$ and $\mathrm{Index}(p, f) \leqslant \mathrm{len}\, f$.

(44) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and there exists a natural number i such that $1 \leqslant i$ and $i + 1 \leqslant \mathrm{len}\, f$ and $p \in \mathcal{L}(f, i)$ and $p \neq f(\mathrm{len}\, f)$. Then $\mathrm{Index}(p, f) < \mathrm{len}\, f$.

(45) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and given i_1. Suppose that
  (i) f is a special sequence,
  (ii) there exists a natural number i such that $1 \leqslant i$ and $i + 1 \leqslant \mathrm{len}\, f$ and $p \in \mathcal{L}(f, i)$, and

(iii) $1 \leqslant i_1$ and $i_1+1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i_1)$ and $p \neq f(i_1+1)$ or $i_1 = \operatorname{len} f$ and $p = f(\operatorname{len} f)$.

Then $i_1 = \operatorname{Index}(p, f)$.

(46) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and there exists a natural number i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i)$ and $p = f(\operatorname{len} f)$. Then $\operatorname{Index}(p, f) = \operatorname{len} f$.

(47) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and given i_1. If f is a special sequence and $1 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} f$ and $p = f(i_1)$, then $\operatorname{Index}(p, f) = i_1$.

(48) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and given i_1. Suppose f is a special sequence and $1 \leqslant i_1$ and $i_1 + 1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i_1)$. Then $i_1 = \operatorname{Index}(p, f)$ or $i_1 + 1 = \operatorname{Index}(p, f)$.

Let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. We say that g is a special sequence joining p_1, p_2 if and only if:

(Def. 3) g is a special sequence and $g(1) = p_1$ and $g(\operatorname{len} g) = p_2$.

One can prove the following propositions:

(49) Let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $1 \leqslant \operatorname{len} g$ and g is a special sequence joining p_1, p_2. Then $\operatorname{Rev}(g)$ is a special sequence joining p_2, p_1.

(50) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and given j. Suppose that

(i) f is a special sequence,

(ii) there exists a natural number i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i)$,

(iii) $p \neq f(\operatorname{len} f)$,

(iv) $g = \langle p \rangle \frown \operatorname{mid}(f, \operatorname{Index}(p, f) + 1, \operatorname{len} f)$,

(v) $1 \leqslant j$, and

(vi) $j + 1 \leqslant \operatorname{len} g$.

Then $\mathcal{L}(g, j) \subseteq \mathcal{L}(f, (\operatorname{Index}(p, f) + j) -' 1)$.

(51) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) f is a special sequence,

(ii) there exists a natural number i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i)$,

(iii) $p \neq f(\operatorname{len} f)$, and

(iv) $g = \langle p \rangle \frown \operatorname{mid}(f, \operatorname{Index}(p, f) + 1, \operatorname{len} f)$.

Then g is a special sequence joining p, $\pi_{\operatorname{len} f} f$.

(52) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and given j. Suppose that

(i) f is a special sequence,

(ii) there exists a natural number i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i)$,

(iii) $1 \leqslant j$,
(iv) $j + 1 \leqslant \operatorname{len} g$,
(v) if $p \neq f(\mathrm{Index}(p, f))$, then $g = (\mathrm{mid}(f, 1, \mathrm{Index}(p, f))) \frown \langle p \rangle$, and
(vi) if $p = f(\mathrm{Index}(p, f))$, then $g = \mathrm{mid}(f, 1, \mathrm{Index}(p, f))$.

Then $\mathcal{L}(g, j) \subseteq \mathcal{L}(f, j)$.

(53) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that
(i) f is a special sequence,
(ii) there exists a natural number i such that $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i)$,
(iii) $p \neq f(1)$,
(iv) if $p \neq f(\mathrm{Index}(p, f))$, then $g = (\mathrm{mid}(f, 1, \mathrm{Index}(p, f))) \frown \langle p \rangle$, and
(v) if $p = f(\mathrm{Index}(p, f))$, then $g = \mathrm{mid}(f, 1, \mathrm{Index}(p, f))$.

Then g is a special sequence joining $\pi_1 f$, p.

4. Left and Right Cutting Functions for Finite Sequences in $\mathcal{E}^2_{\mathrm{T}}$

Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\rfloor p, f$ yielding a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 4) $\rfloor p, f = \langle p \rangle \frown \mathrm{mid}(f, \mathrm{Index}(p, f) + 1, \operatorname{len} f)$.

The functor $\lfloor f, p$ yields a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 5)(i) $\lfloor f, p = (\mathrm{mid}(f, 1, \mathrm{Index}(p, f))) \frown \langle p \rangle$ if $p \neq f(\mathrm{Index}(p, f))$,
(ii) $\lfloor f, p = \mathrm{mid}(f, 1, \mathrm{Index}(p, f))$, otherwise.

Next we state four propositions:

(54) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\operatorname{len} f)$. Then $(\rfloor p, f)(1) = p$ and for every i such that $1 < i$ and $i \leqslant (\operatorname{len} f - \mathrm{Index}(p, f)) + 1$ holds $(\rfloor p, f)(i) = f((\mathrm{Index}(p, f) + i) - 1)$.

(55) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $(\lfloor f, p)(\operatorname{len} \lfloor f, p) = p$ and for every i such that $1 < i$ and $i \leqslant \mathrm{Index}(p, f)$ holds $(\lfloor f, p)(i) = f(i)$.

(56) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\operatorname{len} f)$, then $\operatorname{len} \rfloor p, f = (\operatorname{len} f - \mathrm{Index}(p, f)) + 1$.

(57) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$ such that f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$. Then
(i) if $p \neq f(\mathrm{Index}(p, f))$, then $\operatorname{len} \lfloor f, p = \mathrm{Index}(p, f) + 1$, and
(ii) if $p = f(\mathrm{Index}(p, f))$, then $\operatorname{len} \lfloor f, p = \mathrm{Index}(p, f)$.

Let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. The predicate $\mathrm{LE}(q_1, q_2, p_1, p_2)$ is defined by the conditions (Def. 6).

(Def. 6)(i) $q_1 \in \mathcal{L}(p_1, p_2)$,

(ii) $q_2 \in \mathcal{L}(p_1, p_2)$, and

(iii) for all real numbers r_1, r_2 such that $0 \leqslant r_1$ and $r_1 \leqslant 1$ and $q_1 = (1 - r_1) \cdot p_1 + r_1 \cdot p_2$ and $0 \leqslant r_2$ and $r_2 \leqslant 1$ and $q_2 = (1 - r_2) \cdot p_1 + r_2 \cdot p_2$ holds $r_1 \leqslant r_2$.

Let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. The predicate $\mathrm{LT}(q_1, q_2, p_1, p_2)$ is defined as follows:

(Def. 7) $\mathrm{LE}(q_1, q_2, p_1, p_2)$ and $q_1 \neq q_2$.

Next we state several propositions:

(58) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{LT}(q_1, q_2, p_1, p_2)$ holds $\mathrm{LE}(q_1, q_2, p_1, p_2)$.

(59) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{LE}(q_1, q_2, p_1, p_2)$ and $\mathrm{LE}(q_2, q_1, p_1, p_2)$ holds $q_1 = q_2$.

(60) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $q_1 \in \mathcal{L}(p_1, p_2)$ and $q_2 \in \mathcal{L}(p_1, p_2)$ and $p_1 \neq p_2$ holds $\mathrm{LE}(q_1, q_2, p_1, p_2)$ or $\mathrm{LT}(q_2, q_1, p_1, p_2)$ but $\mathrm{LE}(q_1, q_2, p_1, p_2)$ but $\mathrm{LT}(q_2, q_1, p_1, p_2)$.

(61) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q, p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $\mathrm{Index}(p, f) < \mathrm{Index}(q, f)$, then $q \in \widetilde{\mathcal{L}}(\rfloor p, f)$.

(62) For all points p, q, p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{LE}(p, q, p_1, p_2)$ holds $q \in \mathcal{L}(p, p_2)$ and $p \in \mathcal{L}(p_1, q)$.

(63) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q, p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$ and $\mathrm{Index}(p, f) = \mathrm{Index}(q, f)$ and $\mathrm{LE}(p, q, \pi_{\mathrm{Index}(p,f)} f, \pi_{\mathrm{Index}(p,f)+1} f)$. Then $q \in \widetilde{\mathcal{L}}(\rfloor p, f)$.

5. Cutting Both Sides of a Finite Sequence and a Discussion of Speciality of Sequences in $\mathcal{E}^2_{\mathrm{T}}$

Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\rfloor\lfloor p, f, q$ yielding a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 8)(i) $\rfloor\lfloor p, f, q = \lfloor\rfloor p, f, q$ if $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $\mathrm{Index}(p, f) < \mathrm{Index}(q, f)$ or $\mathrm{Index}(p, f) = \mathrm{Index}(q, f)$ and $\mathrm{LE}(p, q, \pi_{\mathrm{Index}(p,f)} f, \pi_{\mathrm{Index}(p,f)+1} f)$,

(ii) $\rfloor\lfloor p, f, q = \mathrm{Rev}(\lfloor\rfloor q, f, p)$, otherwise.

The following propositions are true:

(64) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$. Then $\rfloor p, f$ is a special sequence joining p, $\pi_{\mathrm{len}\, f} f$.

(65) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\operatorname{len} f)$. Then $\rfloor p, f$ is a special sequence.

(66) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $\lfloor f, p$ is a special sequence joining $\pi_1 f$, p.

(67) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $\lfloor f, p$ is a special sequence.

(68) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p, q be points of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$. Then $\rfloor \lfloor p, f, q$ is a special sequence joining p, q.

(69) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p, q be points of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$. Then $\rfloor \lfloor p, f, q$ is a special sequence.

(70) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $f \frown \operatorname{mid}(g, 2, \operatorname{len} g)$ is a special sequence.

(71) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $f \frown \operatorname{mid}(g, 2, \operatorname{len} g)$ is a special sequence joining $\pi_1 f$, $\pi_{\operatorname{len} g} g$.

(72) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for every natural number n holds $\widetilde{\mathcal{L}}(f_{\restriction n}) \subseteq \widetilde{\mathcal{L}}(f)$.

(73) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\operatorname{len} f)$ and f is a special sequence, then $\widetilde{\mathcal{L}}(\rfloor p, f) \subseteq \widetilde{\mathcal{L}}(f)$.

(74) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq f(\operatorname{len} f)$. Then $(\rfloor p, f) \frown \operatorname{mid}(g, 2, \operatorname{len} g)$ is a special sequence joining p, $\pi_{\operatorname{len} g} g$.

(75) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq f(\operatorname{len} f)$. Then $(\rfloor p, f) \frown \operatorname{mid}(g, 2, \operatorname{len} g)$ is a special sequence.

(76) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown g$ is a special sequence.

(77) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown g$ is a special sequence joining $\pi_1 f$, $\pi_{\operatorname{len} g} g$.

(78) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$ and f is a special sequence, then $\widetilde{\mathcal{L}}(\downharpoonleft f, p) \subseteq \widetilde{\mathcal{L}}(f)$.

(79) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(g)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq g(1)$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown \downharpoonleft g, p$ is a special sequence joining $\pi_1 f$, p.

(80) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(g)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq g(1)$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown \downharpoonleft g, p$ is a special sequence.

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Conjugate Sequences, Bounded Complex Sequences and Convergent Complex Sequences

Adam Naumowicz
Warsaw University
Białystok

Summary. This article is a continuation of [1].It is divided into five sections. The first one contains a few useful lemmas. In the second part there is a definition of conjugate sequences and proofs of some basic properties of such sequences. The third segment treats of bounded complex sequences,next one contains description of convergent complex sequences. The last and the biggest part of the article contains proofs of main theorems concerning the theory of bounded and convergent complex sequences.

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The notation and terminology used in this paper have been introduced in the following articles: [4], [6], [5], [7], [2], [8], [3], and [1].

1. Preliminaries

We adopt the following convention: n, m denote natural numbers, r, p, g denote elements of $\mathbb{C}$, and s, s', s_1 denote complex sequences.

The following propositions are true:

(1) If $g \neq 0_{\mathbb{C}}$ and $r \neq 0_{\mathbb{C}}$, then $|g^{-1} - r^{-1}| = \frac{|g-r|}{|g|\cdot|r|}$.

(2) For every n there exists a real number r such that $0 < r$ and for every m such that $m \leqslant n$ holds $|s(m)| < r$.

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2. Conjugate sequences

Let us consider s. The functor s^* yields a complex sequence and is defined by:

(Def. 1) For every n holds $s^*(n) = s(n)^*$.

We now state several propositions:

(3) If s is non-zero, then s^* is non-zero.

(4) $(r\,s)^* = r^*\,s^*$.

(5) $(s\,s')^* = s^*\,s'^*$.

(6) If s is non-zero, then $(s^*)^{-1} = (s^{-1})^*$.

(7) If s is non-zero, then $(\frac{s'}{s})^* = \frac{s'^*}{s^*}$.

3. Bounded complex sequences

Let us consider s. We say that s is bounded if and only if:

(Def. 2) There exists a real number r such that for every n holds $|s(n)| < r$.

Let us observe that there exists a complex sequence which is bounded.

Next we state the proposition

(8) s is bounded iff there exists a real number r such that $0 < r$ and for every n holds $|s(n)| < r$.

4. Convergent complex sequences

Let us consider s. We say that s is convergent if and only if:

(Def. 3) There exists g such that for every real number p such that $0 < p$ there exists n such that for every m such that $n \leqslant m$ holds $|s(m) - g| < p$.

Let us consider s. Let us assume that s is convergent. The functor $\lim s$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 4) For every real number p such that $0 < p$ there exists n such that for every m such that $n \leqslant m$ holds $|s(m) - \lim s| < p$.

One can prove the following two propositions:

(9) If there exists g such that for every natural number n holds $s(n) = g$, then s is convergent.

(10) For every g such that for every natural number n holds $s(n) = g$ holds $\lim s = g$.

Let us observe that there exists a complex sequence which is convergent.

Let s be a convergent complex sequence. Observe that $|s|$ is convergent.

One can prove the following proposition

(11) If s is convergent, then $\lim |s| = |\lim s|$.

Let s be a convergent complex sequence. Observe that s^* is convergent.

We now state the proposition

(12) If s is convergent, then $\lim(s^*) = (\lim s)^*$.

5. Main theorems

The following propositions are true:

(13) If s is convergent and s' is convergent, then $s + s'$ is convergent.

(14) If s is convergent and s' is convergent, then $\lim(s + s') = \lim s + \lim s'$.

(15) If s is convergent and s' is convergent, then $\lim |s + s'| = |\lim s + \lim s'|$.

(16) If s is convergent and s' is convergent, then $\lim((s + s')^*) = (\lim s)^* + (\lim s')^*$.

(17) If s is convergent, then $r\, s$ is convergent.

(18) If s is convergent, then $\lim(r\, s) = r \cdot \lim s$.

(19) If s is convergent, then $\lim |r\, s| = |r| \cdot |\lim s|$.

(20) If s is convergent, then $\lim((r\, s)^*) = r^* \cdot (\lim s)^*$.

(21) If s is convergent, then $-s$ is convergent.

(22) If s is convergent, then $\lim(-s) = -\lim s$.

(23) If s is convergent, then $\lim |-s| = |\lim s|$.

(24) If s is convergent, then $\lim((-s)^*) = -(\lim s)^*$.

(25) If s is convergent and s' is convergent, then $s - s'$ is convergent.

(26) If s is convergent and s' is convergent, then $\lim(s - s') = \lim s - \lim s'$.

(27) If s is convergent and s' is convergent, then $\lim |s - s'| = |\lim s - \lim s'|$.

(28) If s is convergent and s' is convergent, then $\lim((s - s')^*) = (\lim s)^* - (\lim s')^*$.

Let us mention that every complex sequence which is convergent is also bounded.

Let us note that every complex sequence which is non bounded is also non convergent.

One can prove the following propositions:

(29) If s is a convergent complex sequence and s' is a convergent complex sequence, then $s\, s'$ is convergent.

(30) If s is a convergent complex sequence and s' is a convergent complex sequence, then $\lim(s\, s') = \lim s \cdot \lim s'$.

(31) If s is convergent and s' is convergent, then $\lim |s\, s'| = |\lim s| \cdot |\lim s'|$.

(32) If s is convergent and s' is convergent, then $\lim((s\, s')^*) = (\lim s)^* \cdot (\lim s')^*$.

(33) If s is convergent, then if $\lim s \neq 0_{\mathbb{C}}$, then there exists n such that for every m such that $n \leqslant m$ holds $\frac{|\lim s|}{2} < |s(m)|$.

(34) If s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then s^{-1} is convergent.

(35) If s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim(s^{-1}) = (\lim s)^{-1}$.

(36) If s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim |s^{-1}| = |\lim s|^{-1}$.

(37) If s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim((s^{-1})^*) = ((\lim s)^*)^{-1}$.

(38) If s' is convergent and s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\frac{s'}{s}$ is convergent.

(39) If s' is convergent and s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim(\frac{s'}{s}) = \frac{\lim s'}{\lim s}$.

(40) If s' is convergent and s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim |\frac{s'}{s}| = \frac{|\lim s'|}{|\lim s|}$.

(41) If s' is convergent and s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim((\frac{s'}{s})^*) = \frac{(\lim s')^*}{(\lim s)^*}$.

(42) If s is convergent and s_1 is bounded and $\lim s = 0_{\mathbb{C}}$, then $s\, s_1$ is convergent.

(43) If s is convergent and s_1 is bounded and $\lim s = 0_{\mathbb{C}}$, then $\lim(s\, s_1) = 0_{\mathbb{C}}$.

(44) If s is convergent and s_1 is bounded and $\lim s = 0_{\mathbb{C}}$, then $\lim |s\, s_1| = 0$.

(45) If s is convergent and s_1 is bounded and $\lim s = 0_{\mathbb{C}}$, then $\lim((s\, s_1)^*) = 0_{\mathbb{C}}$.

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On the Topological Properties of Meet-Continuous Lattices[1]

Artur Korniłowicz
Warsaw University
Białystok

Summary. This work is continuation of formalization of [12]. Proposition 4.4 from Chapter 0 is proved.

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The terminology and notation used in this paper are introduced in the following papers: [30], [24], [25], [27], [31], [32], [33], [6], [11], [21], [15], [7], [23], [34], [10], [29], [9], [5], [4], [28], [22], [1], [13], [16], [2], [3], [14], [35], [8], [17], [20], [18], [19], and [26].

1. Preliminaries

Let L be a non empty relational structure. One can check that id_L is monotone.

Let S, T be non empty relational structures and let f be a map from S into T. Let us observe that f is antitone if and only if:

(Def. 1) For all elements x, y of S such that $x \leqslant y$ holds $f(x) \geqslant f(y)$.

Next we state several propositions:

(1) Let S, T be relational structures, K, L be non empty relational structures, f be a map from S into T, and g be a map from K into L. Suppose that

(i) the relational structure of S = the relational structure of K,

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(ii) the relational structure of $T =$ the relational structure of L,
(iii) $f = g$, and
(iv) f is monotone.
Then g is monotone.

(2) Let S, T be relational structures, K, L be non empty relational structures, f be a map from S into T, and g be a map from K into L. Suppose that
(i) the relational structure of $S =$ the relational structure of K,
(ii) the relational structure of $T =$ the relational structure of L,
(iii) $f = g$, and
(iv) f is antitone.
Then g is antitone.

(3) Let A, B be 1-sorted structures, F be a family of subsets of A, and G be a family of subsets of B. Suppose the carrier of $A =$ the carrier of B and $F = G$ and F is a cover of A. Then G is a cover of B.

(4) For every antisymmetric reflexive relational structure L with l.u.b.'s and for every element x of L holds $\uparrow x = \{x\} \sqcup \Omega_L$.

(5) For every antisymmetric reflexive relational structure L with g.l.b.'s and for every element x of L holds $\downarrow x = \{x\} \sqcap \Omega_L$.

(6) For every antisymmetric reflexive relational structure L with g.l.b.'s and for every element y of L holds $(y \sqcap \square)^{\circ} \uparrow y = \{y\}$.

(7) For every antisymmetric reflexive relational structure L with g.l.b.'s and for every element x of L holds $(x \sqcap \square)^{-1}(\{x\}) = \uparrow x$.

(8) For every non empty 1-sorted structure T holds every non empty net structure N over T is eventually in rng (the mapping of N).

Let L be a non empty reflexive relational structure, let D be a non empty directed subset of L, and let n be a function from D into the carrier of L. One can verify that $\langle D, (\text{the internal relation of } L) \,|^2\, D, n\rangle$ is directed.

Let L be a non empty reflexive transitive relational structure, let D be a non empty directed subset of L, and let n be a function from D into the carrier of L. One can check that $\langle D, (\text{the internal relation of } L) \,|^2\, D, n\rangle$ is transitive.

The following propositions are true:

(9) For every non empty reflexive transitive relational structure L such that for every element x of L and for every net N in L such that N is eventually-directed holds $x \sqcap \sup N = \sup\{x\} \sqcap \operatorname{rng} \operatorname{netmap}(N, L)$ holds L satisfies MC.

(10) Let L be a non empty relational structure, a be an element of L, and N be a net in L. Then $a \sqcap N$ is a net in L.

Let L be a non empty relational structure, let x be an element of L, and let N be a net in L. Then $x \sqcap N$ is a strict net in L.

Let L be a non empty relational structure, let x be an element of L, and let N be a non empty reflexive net structure over L. Observe that $x \sqcap N$ is reflexive.

Let L be a non empty relational structure, let x be an element of L, and let N be a non empty antisymmetric net structure over L. Note that $x \sqcap N$ is antisymmetric.

Let L be a non empty relational structure, let x be an element of L, and let N be a non empty transitive net structure over L. Note that $x \sqcap N$ is transitive.

Let L be a non empty relational structure, let J be a set, and let f be a function from J into the carrier of L. Observe that $\mathrm{FinSups}(f)$ is transitive.

2. The Operations Defined on Nets

Let L be a non empty relational structure and let N be a net structure over L. The functor $\inf N$ yielding an element of L is defined as follows:

(Def. 2) $\inf N = \mathrm{Inf}(\text{the mapping of } N)$.

Let L be a relational structure and let N be a net structure over L. We say that $\sup N$ exists if and only if:

(Def. 3) $\mathrm{Sup}\,\mathrm{rng}\,(\text{the mapping of } N)$ exists in L.

We say that $\inf N$ exists if and only if:

(Def. 4) $\mathrm{Inf}\,\mathrm{rng}\,(\text{the mapping of } N)$ exists in L.

Let L be a relational structure. The functor $\langle L; \mathrm{id}\rangle$ yields a strict net structure over L and is defined by:

(Def. 5) The relational structure of $\langle L; \mathrm{id}\rangle$ = the relational structure of L and the mapping of $\langle L; \mathrm{id}\rangle = \mathrm{id}_L$.

Let L be a non empty relational structure. Observe that $\langle L; \mathrm{id}\rangle$ is non empty.

Let L be a reflexive relational structure. One can check that $\langle L; \mathrm{id}\rangle$ is reflexive.

Let L be an antisymmetric relational structure. Note that $\langle L; \mathrm{id}\rangle$ is antisymmetric.

Let L be a transitive relational structure. Observe that $\langle L; \mathrm{id}\rangle$ is transitive.

Let L be a relational structure with l.u.b.'s. One can verify that $\langle L; \mathrm{id}\rangle$ is directed.

Let L be a directed relational structure. Note that $\langle L; \mathrm{id}\rangle$ is directed.

Let L be a non empty relational structure. One can verify that $\langle L; \mathrm{id}\rangle$ is monotone and eventually-directed.

Let L be a relational structure. The functor $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$ yields a strict net structure over L and is defined by the conditions (Def. 6).

(Def. 6)(i) The carrier of $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$ = the carrier of L,

(ii) the internal relation of $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$ = (the internal relation of $L)^{\smile}$, and

(iii) the mapping of $\langle L^{\mathrm{op}}; \mathrm{id}\rangle = \mathrm{id}_L$.

Next we state the proposition

(11) For every relational structure L holds the relational structure of $L^{\smile}$ = the relational structure of $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$.

Let L be a non empty relational structure. One can check that $\langle L^{\rm op}; {\rm id}\rangle$ is non empty.

Let L be a reflexive relational structure. Observe that $\langle L^{\rm op}; {\rm id}\rangle$ is reflexive.

Let L be an antisymmetric relational structure. Observe that $\langle L^{\rm op}; {\rm id}\rangle$ is antisymmetric.

Let L be a transitive relational structure. Note that $\langle L^{\rm op}; {\rm id}\rangle$ is transitive.

Let L be a relational structure with g.l.b.'s. Note that $\langle L^{\rm op}; {\rm id}\rangle$ is directed.

Let L be a non empty relational structure. Note that $\langle L^{\rm op}; {\rm id}\rangle$ is antitone and eventually-filtered.

Let L be a non empty 1-sorted structure, let N be a non empty net structure over L, and let i be an element of N. The functor $N{\restriction}i$ yields a strict net structure over L and is defined by the conditions (Def. 7).

(Def. 7)(i) For every set x holds $x \in$ the carrier of $N{\restriction}i$ iff there exists an element y of N such that $y = x$ and $i \leqslant y$,

(ii) the internal relation of $N{\restriction}i =$ (the internal relation of N) $|^2$ (the carrier of $N{\restriction}i$), and

(iii) the mapping of $N{\restriction}i =$ (the mapping of N)${\restriction}$(the carrier of $N{\restriction}i$).

We now state three propositions:

(12) Let L be a non empty 1-sorted structure, N be a non empty net structure over L, and i be an element of N. Then the carrier of $N{\restriction}i = \{y, y$ ranges over elements of N: $i \leqslant y\}$.

(13) Let L be a non empty 1-sorted structure, N be a non empty net structure over L, and i be an element of N. Then the carrier of $N{\restriction}i \subseteq$ the carrier of N.

(14) Let L be a non empty 1-sorted structure, N be a non empty net structure over L, and i be an element of N. Then $N{\restriction}i$ is a full structure of a subnet of N.

Let L be a non empty 1-sorted structure, let N be a non empty reflexive net structure over L, and let i be an element of N. Note that $N{\restriction}i$ is non empty and reflexive.

Let L be a non empty 1-sorted structure, let N be a non empty directed net structure over L, and let i be an element of N. Note that $N{\restriction}i$ is non empty.

Let L be a non empty 1-sorted structure, let N be a non empty reflexive antisymmetric net structure over L, and let i be an element of N. Observe that $N{\restriction}i$ is antisymmetric.

Let L be a non empty 1-sorted structure, let N be a non empty directed antisymmetric net structure over L, and let i be an element of N. Note that $N{\restriction}i$ is antisymmetric.

Let L be a non empty 1-sorted structure, let N be a non empty reflexive transitive net structure over L, and let i be an element of N. One can verify that $N{\restriction}i$ is transitive.

Let L be a non empty 1-sorted structure, let N be a net in L, and let i be an element of N. Note that $N{\restriction}i$ is transitive and directed.

Next we state three propositions:

(15) Let L be a non empty 1-sorted structure, N be a non empty reflexive net structure over L, i, x be elements of N, and x_1 be an element of $N \restriction i$. If $x = x_1$, then $N(x) = (N \restriction i)(x_1)$.

(16) Let L be a non empty 1-sorted structure, N be a non empty directed net structure over L, i, x be elements of N, and x_1 be an element of $N \restriction i$. If $x = x_1$, then $N(x) = (N \restriction i)(x_1)$.

(17) Let L be a non empty 1-sorted structure, N be a net in L, and i be an element of N. Then $N \restriction i$ is a subnet of N.

Let T be a non empty 1-sorted structure and let N be a net in T. Observe that there exists a subnet of N which is strict.

Let L be a non empty 1-sorted structure, let N be a net in L, and let i be an element of N. Then $N \restriction i$ is a strict subnet of N.

Let S be a non empty 1-sorted structure, let T be a 1-sorted structure, let f be a map from S into T, and let N be a net structure over S. The functor $f \cdot N$ yielding a strict net structure over T is defined by the conditions (Def. 8).

(Def. 8)(i) The relational structure of $f \cdot N$ = the relational structure of N, and

(ii) the mapping of $f \cdot N = f \cdot$ the mapping of N.

Let S be a non empty 1-sorted structure, let T be a 1-sorted structure, let f be a map from S into T, and let N be a non empty net structure over S. One can verify that $f \cdot N$ is non empty.

Let S be a non empty 1-sorted structure, let T be a 1-sorted structure, let f be a map from S into T, and let N be a reflexive net structure over S. Observe that $f \cdot N$ is reflexive.

Let S be a non empty 1-sorted structure, let T be a 1-sorted structure, let f be a map from S into T, and let N be an antisymmetric net structure over S. Observe that $f \cdot N$ is antisymmetric.

Let S be a non empty 1-sorted structure, let T be a 1-sorted structure, let f be a map from S into T, and let N be a transitive net structure over S. Note that $f \cdot N$ is transitive.

Let S be a non empty 1-sorted structure, let T be a 1-sorted structure, let f be a map from S into T, and let N be a directed net structure over S. Note that $f \cdot N$ is directed.

One can prove the following proposition

(18) Let L be a non empty relational structure, N be a non empty net structure over L, and x be an element of L. Then $(x \sqcap \square) \cdot N = x \sqcap N$.

3. The Properties of Topological Spaces

The following two propositions are true:

(19) Let S, T be topological structures, F be a family of subsets of S, and G be a family of subsets of T. Suppose the topological structure of S = the topological structure of T and $F = G$ and F is open. Then G is open.

(20) Let S, T be topological structures, F be a family of subsets of S, and G be a family of subsets of T. Suppose the topological structure of $S =$ the topological structure of T and $F = G$ and F is closed. Then G is closed.

Let a be a set. Note that $\{a\}_{\mathrm{top}}$ is discrete.

We consider FR-structures as extensions of topological structure and relational structure as systems

⟨ a carrier, a internal relation, a topology ⟩,

where the carrier is a set, the internal relation is a binary relation on the carrier, and the topology is a family of subsets of the carrier.

Let A be a non empty set, let R be a relation between A and A, and let T be a family of subsets of A. Note that $\langle A, R, T \rangle$ is non empty.

Let x be a set, let R be a binary relation on $\{x\}$, and let T be a family of subsets of $\{x\}$. Note that $\langle \{x\}, R, T \rangle$ is trivial.

Let X be a set, let O be an order in X, and let T be a family of subsets of X. Observe that $\langle X, O, T \rangle$ is reflexive transitive and antisymmetric.

Let us observe that there exists a FR-structure which is trivial, reflexive, non empty, discrete, strict, and finite.

A TopLattice is a reflexive transitive antisymmetric topological space-like FR-structure with g.l.b.'s and l.u.b.'s.

Let us observe that there exists a non empty TopLattice which is strict, trivial, discrete, finite, compact, and Hausdorff.

Let T be a Hausdorff non empty topological space. One can check that every non empty subspace of T is Hausdorff.

One can prove the following propositions:

(21) For every non empty topological space T and for every point p of T holds every element of the open neighbourhoods of p is a neighbourhood of p.

(22) Let T be a non empty topological space, p be a point of T, and A, B be elements of the open neighbourhoods of p. Then $A \cap B$ is an element of the open neighbourhoods of p.

(23) Let T be a non empty topological space, p be a point of T, and A, B be elements of the open neighbourhoods of p. Then $A \cup B$ is an element of the open neighbourhoods of p.

(24) Let T be a non empty topological space, p be an element of the carrier of T, and N be a net in T. Suppose $p \in \operatorname{Lim} N$. Let S be a subset of the carrier of T. If $S = \operatorname{rng}$ (the mapping of N), then $p \in \overline{S}$.

(25) Let T be a Hausdorff non empty TopLattice, N be a convergent net in T, and f be a map from T into T. If f is continuous, then $f(\lim N) \in \operatorname{Lim}(f \cdot N)$.

(26) Let T be a Hausdorff non empty TopLattice, N be a convergent net in T, and x be an element of T. If $x \sqcap \square$ is continuous, then $x \sqcap \lim N \in \operatorname{Lim}(x \sqcap N)$.

(27) Let S be a Hausdorff non empty TopLattice and x be an element of S. If for every element a of S holds $a \sqcap \square$ is continuous, then $\uparrow x$ is closed.

(28) Let S be a compact Hausdorff non empty TopLattice and x be an element of S. If for every element b of S holds $b \sqcap \square$ is continuous, then $\downarrow x$ is closed.

4. The Cluster Points of Nets

Let T be a TopLattice, let N be a non empty net structure over T, and let p be a point of T. We say that p is a cluster point of N if and only if:

(Def. 9) For every neighbourhood O of p holds N is often in O.

Next we state several propositions:

(29) Let L be a non empty TopLattice, N be a net in L, and c be a point of L. If $c \in \operatorname{Lim} N$, then c is a cluster point of N.

(30) Let T be a compact Hausdorff non empty TopLattice and N be a net in T. Then there exists a point c of T such that c is a cluster point of N.

(31) Let L be a non empty TopLattice, N be a net in L, M be a subnet of N, and c be a point of L. If c is a cluster point of M, then c is a cluster point of N.

(32) Let T be a non empty TopLattice, N be a net in T, and x be a point of T. Suppose x is a cluster point of N. Then there exists a subnet M of N such that $x \in \operatorname{Lim} M$.

(33) Let L be a compact Hausdorff non empty TopLattice and N be a net in L. Suppose that for all points c, d of L such that c is a cluster point of N and d is a cluster point of N holds $c = d$. Let s be a point of L. If s is a cluster point of N, then $s \in \operatorname{Lim} N$.

(34) Let S be a non empty TopLattice, c be a point of S, N be a net in S, and A be a subset of S. Suppose c is a cluster point of N and A is closed and $\operatorname{rng}(\text{the mapping of } N) \subseteq A$. Then $c \in A$.

(35) Let S be a compact Hausdorff non empty TopLattice, c be a point of S, and N be a net in S. Suppose for every element x of S holds $x \sqcap \square$ is continuous and N is eventually-directed and c is a cluster point of N. Then $c = \sup N$.

(36) Let S be a compact Hausdorff non empty TopLattice, c be a point of S, and N be a net in S. Suppose for every element x of S holds $x \sqcap \square$ is continuous and N is eventually-filtered and c is a cluster point of N. Then $c = \inf N$.

5. On The Topological Properties of Meet-Continuous Lattices

Next we state several propositions:

(37) Let S be a Hausdorff non empty TopLattice. Suppose that
 (i) for every net N in S such that N is eventually-directed holds sup N exists and $\sup N \in \operatorname{Lim} N$, and
 (ii) for every element x of S holds $x \sqcap \square$ is continuous.
 Then S is meet-continuous.

(38) Let S be a compact Hausdorff non empty TopLattice. Suppose that for every element x of S holds $x \sqcap \square$ is continuous. Let N be a net in S. If N is eventually-directed, then sup N exists and $\sup N \in \operatorname{Lim} N$.

(39) Let S be a compact Hausdorff non empty TopLattice. Suppose that for every element x of S holds $x \sqcap \square$ is continuous. Let N be a net in S. If N is eventually-filtered, then inf N exists and $\inf N \in \operatorname{Lim} N$.

(40) Let S be a compact Hausdorff non empty TopLattice. If for every element x of S holds $x \sqcap \square$ is continuous, then S is bounded.

(41) Let S be a compact Hausdorff non empty TopLattice. Suppose that for every element x of S holds $x \sqcap \square$ is continuous. Then S is meet-continuous.

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Institution of Many Sorted Algebras. Part I: Signature Reduct of an Algebra

Grzegorz Bancerek
Warsaw University
Białystok

Summary. In the paper the notation necessary to construct the institution of many sorted algebras is introduced.

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The papers [23], [27], [16], [1], [28], [14], [9], [13], [2], [26], [17], [3], [4], [10], [6], [11], [20], [24], [25], [15], [12], [21], [19], [5], [22], [7], [18], and [8] provide the terminology and notation for this paper.

1. Preliminaries

One can prove the following propositions:

(1) Let I be a set, f be a function, and F, G be many sorted functions indexed by I. If $\operatorname{rng} f \subseteq I$, then $(G \circ F) \cdot f = (G \cdot f) \circ (F \cdot f)$.

(2) Let S be a non empty non void many sorted signature, o be an operation symbol of S, V be a non-empty many sorted set indexed by the carrier of S, and x be a set. Then x is an argument sequence of $\mathrm{Sym}(o, V)$ if and only if x is an element of $\mathrm{Args}(o, \mathrm{Free}(V))$.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, and let o be an operation symbol of S. Note that every element of $\mathrm{Args}(o, \mathrm{Free}(V))$ is decorated tree yielding.

Next we state two propositions:

(3) Let S be a non empty non void many sorted signature and A_1, A_2 be algebras over S. Suppose the sorts of A_1 are transformable to the sorts of A_2. Let o be an operation symbol of S. If $\mathrm{Args}(o, A_1) \neq \emptyset$, then $\mathrm{Args}(o, A_2) \neq \emptyset$.

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(4) Let S be a non empty non void many sorted signature, o be an operation symbol of S, V be a non-empty many sorted set indexed by the carrier of S, and x be an element of $\mathrm{Args}(o, \mathrm{Free}(V))$. Then $(\mathrm{Den}(o, \mathrm{Free}(V)))(x) = \langle o$, the carrier of $S\rangle$-tree(x).

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S. One can check that the algebra of A is non-empty.

Next we state three propositions:

(5) Let S be a non empty non void many sorted signature and A, B be algebras over S. Suppose the algebra of $A =$ the algebra of B. Let o be an operation symbol of S. Then $\mathrm{Den}(o, A) = \mathrm{Den}(o, B)$.

(6) Let S be a non empty non void many sorted signature and A_1, A_2, B_1, B_2 be algebras over S. Suppose the algebra of $A_1 =$ the algebra of B_1 and the algebra of $A_2 =$ the algebra of B_2. Let f be a many sorted function from A_1 into A_2 and g be a many sorted function from B_1 into B_2. Suppose $f = g$. Let o be an operation symbol of S. Suppose $\mathrm{Args}(o, A_1) \neq \emptyset$ and $\mathrm{Args}(o, A_2) \neq \emptyset$. Let x be an element of $\mathrm{Args}(o, A_1)$ and y be an element of $\mathrm{Args}(o, B_1)$. If $x = y$, then $f\#x = g\#y$.

(7) Let S be a non empty non void many sorted signature and A_1, A_2, B_1, B_2 be algebras over S. Suppose that

(i) the algebra of $A_1 =$ the algebra of B_1,

(ii) the algebra of $A_2 =$ the algebra of B_2, and

(iii) the sorts of A_1 are transformable to the sorts of A_2.

Let h be a many sorted function from A_1 into A_2. Suppose h is a homomorphism of A_1 into A_2. Then there exists a many sorted function h' from B_1 into B_2 such that $h' = h$ and h' is a homomorphism of B_1 into B_2.

Let S be a many sorted signature. We say that S is feasible if and only if:

(Def. 1) If the carrier of $S = \emptyset$, then the operation symbols of $S = \emptyset$.

The following proposition is true

(8) Let S be a many sorted signature. Then S is feasible if and only if dom (the result sort of S) $=$ the operation symbols of S.

One can verify the following observations:

* every many sorted signature which is non empty is also feasible,
* every many sorted signature which is void is also feasible,
* every many sorted signature which is empty and feasible is also void, and
* every many sorted signature which is non void and feasible is also non empty.

Let us note that there exists a many sorted signature which is non void and non empty.

One can prove the following propositions:

(9) Let S be a feasible many sorted signature. Then $\mathrm{id}_{\text{the carrier of } S}$ and $\mathrm{id}_{\text{the operation symbols of } S}$ form morphism between S and S.

(10) Let S_1, S_2 be many sorted signatures and f, g be functions. Suppose f and g form morphism between S_1 and S_2. Then
(i) f is a function from the carrier of S_1 into the carrier of S_2, and
(ii) g is a function from the operation symbols of S_1 into the operation symbols of S_2.

2. Subsignatures

Let S be a feasible many sorted signature. A many sorted signature is said to be a subsignature of S if:

(Def. 2) $\mathrm{id}_{\text{the carrier of it}}$ and $\mathrm{id}_{\text{the operation symbols of it}}$ form morphism between it and S.

We now state the proposition

(11) Let S be a feasible many sorted signature and T be a subsignature of S. Then the carrier of $T \subseteq$ the carrier of S and the operation symbols of $T \subseteq$ the operation symbols of S.

Let S be a feasible many sorted signature. Note that every subsignature of S is feasible.

Next we state several propositions:

(12) Let S be a feasible many sorted signature and T be a subsignature of S. Then the result sort of $T \subseteq$ the result sort of S and the arity of $T \subseteq$ the arity of S.

(13) Let S be a feasible many sorted signature and T be a subsignature of S. Then
(i) the arity of $T =$ (the arity of S)$\restriction$(the operation symbols of T), and
(ii) the result sort of $T =$ (the result sort of S)$\restriction$(the operation symbols of T).

(14) Let S, T be feasible many sorted signatures. Suppose that
(i) the carrier of $T \subseteq$ the carrier of S,
(ii) the arity of $T \subseteq$ the arity of S, and
(iii) the result sort of $T \subseteq$ the result sort of S.
Then T is a subsignature of S.

(15) Let S, T be feasible many sorted signatures. Suppose that
(i) the carrier of $T \subseteq$ the carrier of S,
(ii) the arity of $T =$ (the arity of S)$\restriction$(the operation symbols of T), and
(iii) the result sort of $T =$ (the result sort of S)$\restriction$(the operation symbols of T).
Then T is a subsignature of S.

(16) Every feasible many sorted signature S is a subsignature of S.

(17) For every feasible many sorted signature S_1 and for every subsignature S_2 of S_1 holds every subsignature of S_2 is a subsignature of S_1.

(18) Let S_1 be a feasible many sorted signature and S_2 be a subsignature of S_1. Suppose S_1 is a subsignature of S_2. Then the many sorted signature of S_1 = the many sorted signature of S_2.

Let S be a non empty many sorted signature. Observe that there exists a subsignature of S which is non empty.

Let S be a non void feasible many sorted signature. One can verify that there exists a subsignature of S which is non void.

One can prove the following three propositions:

(19) Let S be a feasible many sorted signature, S' be a subsignature of S, T be a many sorted signature, and f, g be functions. Suppose f and g form morphism between S and T. Then $f\restriction$the carrier of S' and $g\restriction$the operation symbols of S' form morphism between S' and T.

(20) Let S be a many sorted signature, T be a feasible many sorted signature, T' be a subsignature of T, and f, g be functions. Suppose f and g form morphism between S and T'. Then f and g form morphism between S and T.

(21) Let S be a many sorted signature, T be a feasible many sorted signature, T' be a subsignature of T, and f, g be functions. Suppose f and g form morphism between S and T and $\operatorname{rng} f \subseteq$ the carrier of T' and $\operatorname{rng} g \subseteq$ the operation symbols of T'. Then f and g form morphism between S and T'.

3. Signature reducts

Let S_1, S_2 be non empty many sorted signatures, let A be an algebra over S_2, and let f, g be functions. Let us assume that f and g form morphism between S_1 and S_2. The functor $A\restriction_{(f,g)}S_1$ yields a strict algebra over S_1 and is defined by the conditions (Def. 3).

(Def. 3)(i) The sorts of $A\restriction_{(f,g)}S_1$ = (the sorts of A) $\cdot f$, and

(ii) the characteristics of $A\restriction_{(f,g)}S_1$ = (the characteristics of A) $\cdot g$.

Let S_2, S_1 be non empty many sorted signatures and let A be an algebra over S_2. The functor $A\restriction S_1$ yields a strict algebra over S_1 and is defined as follows:

(Def. 4) $A\restriction S_1 = A\restriction_{(\mathrm{id}_{\text{the carrier of } S_1}, \mathrm{id}_{\text{the operation symbols of } S_1})}S_1$.

We now state two propositions:

(22) Let S_1, S_2 be non empty many sorted signatures and A, B be algebras over S_2. Suppose the algebra of A = the algebra of B. Let f, g be functions. If f and g form morphism between S_1 and S_2, then $A\restriction_{(f,g)}S_1 = B\restriction_{(f,g)}S_1$.

(23) Let S_1, S_2 be non empty many sorted signatures, A be a non-empty algebra over S_2, and f, g be functions. If f and g form morphism between S_1 and S_2, then $A\restriction_{(f,g)}S_1$ is non-empty.

Let S_2 be a non empty many sorted signature, let S_1 be a non empty subsignature of S_2, and let A be a non-empty algebra over S_2. Observe that $A\restriction S_1$ is non-empty.

The following propositions are true:

(24) Let S_1, S_2 be non void non empty many sorted signatures and f, g be functions. Suppose f and g form morphism between S_1 and S_2. Let A be an algebra over S_2, o_1 be an operation symbol of S_1, and o_2 be an operation symbol of S_2. If $o_2 = g(o_1)$, then $\mathrm{Den}(o_1, A\restriction_{(f,g)}S_1) = \mathrm{Den}(o_2, A)$.

(25) Let S_1, S_2 be non void non empty many sorted signatures and f, g be functions. Suppose f and g form morphism between S_1 and S_2. Let A be an algebra over S_2, o_1 be an operation symbol of S_1, and o_2 be an operation symbol of S_2. If $o_2 = g(o_1)$, then $\mathrm{Args}(o_2, A) = \mathrm{Args}(o_1, A\restriction_{(f,g)}S_1)$ and $\mathrm{Result}(o_1, A\restriction_{(f,g)}S_1) = \mathrm{Result}(o_2, A)$.

(26) Let S be a non empty many sorted signature and A be an algebra over S. Then $A\restriction_{(\mathrm{id}_{\text{the carrier of } S}, \mathrm{id}_{\text{the operation symbols of } S})}S =$ the algebra of A.

(27) For every non empty many sorted signature S and for every algebra A over S holds $A\restriction S =$ the algebra of A.

(28) Let S_1, S_2, S_3 be non empty many sorted signatures and f_1, g_1 be functions. Suppose f_1 and g_1 form morphism between S_1 and S_2. Let f_2, g_2 be functions. Suppose f_2 and g_2 form morphism between S_2 and S_3. Let A be an algebra over S_3. Then $A\restriction_{(f_2 \cdot f_1, g_2 \cdot g_1)}S_1 = A\restriction_{(f_2,g_2)}S_2\restriction_{(f_1,g_1)}S_1$.

(29) Let S_1 be a non empty feasible many sorted signature, S_2 be a non empty subsignature of S_1, S_3 be a non empty subsignature of S_2, and A be an algebra over S_1. Then $A\restriction S_3 = A\restriction S_2\restriction S_3$.

(30) Let S_1, S_2 be non empty many sorted signatures, f be a function from the carrier of S_1 into the carrier of S_2, and g be a function. Suppose f and g form morphism between S_1 and S_2. Let A_1, A_2 be algebras over S_2 and h be a many sorted function from the sorts of A_1 into the sorts of A_2. Then $h \cdot f$ is a many sorted function from the sorts of $A_1\restriction_{(f,g)}S_1$ into the sorts of $A_2\restriction_{(f,g)}S_1$.

(31) Let S_1 be a non empty many sorted signature, S_2 be a non empty subsignature of S_1, A_1, A_2 be algebras over S_1, and h be a many sorted function from the sorts of A_1 into the sorts of A_2. Then $h\restriction$the carrier of S_2 is a many sorted function from the sorts of $A_1\restriction S_2$ into the sorts of $A_2\restriction S_2$.

(32) Let S_1, S_2 be non empty many sorted signatures and f, g be functions. Suppose f and g form morphism between S_1 and S_2. Let A be an algebra over S_2. Then $\mathrm{id}_{\text{the sorts of } A} \cdot f = \mathrm{id}_{\text{the sorts of } A\restriction_{(f,g)}S_1}$.

(33) Let S_1 be a non empty many sorted signature, S_2 be a non empty subsignature of S_1, and A be an algebra over S_1 Then $\mathrm{id}_{\text{the sorts of } A}\restriction$the carrier of $S_2 = \mathrm{id}_{\text{the sorts of } A\restriction S_2}$.

(34) Let S_1, S_2 be non void non empty many sorted signatures and f, g be functions. Suppose f and g form morphism between S_1 and S_2. Let A, B be algebras over S_2, h_2 be a many sorted function from A into B, and h_1 be a many sorted function from $A\restriction_{(f,g)}S_1$ into $B\restriction_{(f,g)}S_1$. Suppose $h_1 = h_2 \cdot f$. Let o_1 be an operation symbol of S_1 and o_2 be an operation symbol of S_2.

Suppose $o_2 = g(o_1)$ and $\mathrm{Args}(o_2, A) \neq \emptyset$ and $\mathrm{Args}(o_2, B) \neq \emptyset$. Let x_2 be an element of $\mathrm{Args}(o_2, A)$ and x_1 be an element of $\mathrm{Args}(o_1, A{\restriction_{(f,g)}}S_1)$. If $x_2 = x_1$, then $h_1 \# x_1 = h_2 \# x_2$.

(35) Let S, S' be non empty non void many sorted signatures and A_1, A_2 be algebras over S. Suppose the sorts of A_1 are transformable to the sorts of A_2. Let h be a many sorted function from A_1 into A_2. Suppose h is a homomorphism of A_1 into A_2. Let f be a function from the carrier of S' into the carrier of S and g be a function. Suppose f and g form morphism between S' and S. Then there exists a many sorted function h' from $A_1{\restriction_{(f,g)}}S'$ into $A_2{\restriction_{(f,g)}}S'$ such that $h' = h \cdot f$ and h' is a homomorphism of $A_1{\restriction_{(f,g)}}S'$ into $A_2{\restriction_{(f,g)}}S'$.

(36) Let S be a non void feasible many sorted signature, S' be a non void subsignature of S, and A_1, A_2 be algebras over S. Suppose the sorts of A_1 are transformable to the sorts of A_2. Let h be a many sorted function from A_1 into A_2. Suppose h is a homomorphism of A_1 into A_2. Then there exists a many sorted function h' from $A_1{\restriction}S'$ into $A_2{\restriction}S'$ such that $h' = h{\restriction}$the carrier of S' and h' is a homomorphism of $A_1{\restriction}S'$ into $A_2{\restriction}S'$.

(37) Let S, S' be non empty non void many sorted signatures, A be a non-empty algebra over S, f be a function from the carrier of S' into the carrier of S, and g be a function. Suppose f and g form morphism between S' and S. Let B be a non-empty algebra over S'. Suppose $B = A{\restriction_{(f,g)}}S'$. Let s_1, s_2 be sort symbols of S' and t be a function. Suppose t is an elementary translation in B from s_1 into s_2. Then t is an elementary translation in A from $f(s_1)$ into $f(s_2)$.

(38) Let S, S' be non empty non void many sorted signatures, f be a function from the carrier of S' into the carrier of S, and g be a function. Suppose f and g form morphism between S' and S. Let s_1, s_2 be sort symbols of S'. If $\mathrm{TranslRel}(S')$ reduces s_1 to s_2, then $\mathrm{TranslRel}(S)$ reduces $f(s_1)$ to $f(s_2)$.

(39) Let S, S' be non void non empty many sorted signatures, A be a non-empty algebra over S, f be a function from the carrier of S' into the carrier of S, and g be a function. Suppose f and g form morphism between S' and S. Let B be a non-empty algebra over S'. Suppose $B = A{\restriction_{(f,g)}}S'$. Let s_1, s_2 be sort symbols of S'. Suppose $\mathrm{TranslRel}(S')$ reduces s_1 to s_2. Then every translation in B from s_1 into s_2 is a translation in A from $f(s_1)$ into $f(s_2)$.

4. Translating homomorphisms

The scheme *GenFuncEx* concerns a non empty non void many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a non-empty many sorted set $\mathcal{C}$ indexed by the carrier of $\mathcal{A}$, and a binary functor $\mathcal{F}$ yielding a set, and states that:

There exists a many sorted function h from Free($\mathcal{C}$) into $\mathcal{B}$ such that
(i) h is a homomorphism of Free($\mathcal{C}$) into $\mathcal{B}$, and
(ii) for every sort symbol s of $\mathcal{A}$ and for every element x of $\mathcal{C}(s)$ holds $h(s)$(the root tree of $\langle x,\ s\rangle) = \mathcal{F}(x, s)$

provided the parameters meet the following requirement:

- For every sort symbol s of $\mathcal{A}$ and for every element x of $\mathcal{C}(s)$ holds $\mathcal{F}(x, s) \in$ (the sorts of $\mathcal{B})(s)$.

One can prove the following proposition

(40) Let I be a set, A, B be many sorted sets indexed by I, C be a many sorted subset of A, F be a many sorted function from A into B, and i be a set. Suppose $i \in I$. Let f, g be functions. Suppose $f = F(i)$ and $g = (F \restriction C)(i)$. Let x be a set. If $x \in C(i)$, then $g(x) = f(x)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Note that FreeGenerator(X) is non-empty.

Let S_1, S_2 be non empty non void many sorted signatures, let X be a non-empty many sorted set indexed by the carrier of S_2, let f be a function from the carrier of S_1 into the carrier of S_2, and let g be a function. Let us assume that f and g form morphism between S_1 and S_2. The functor hom(f, g, X, S_1, S_2) yields a many sorted function from Free$(X \cdot f)$ into Free$(X)\restriction_{(f,g)}S_1$ and is defined by the conditions (Def. 5).

(Def. 5)(i) hom(f, g, X, S_1, S_2) is a homomorphism of Free$(X \cdot f)$ into Free$(X)\restriction_{(f,g)}S_1$, and

(ii) for every sort symbol s of S_1 and for every element x of $(X \cdot f)(s)$ holds (hom$(f, g, X, S_1, S_2))(s)$(the root tree of $\langle x,\ s\rangle$) = the root tree of $\langle x,\ f(s)\rangle$.

We now state several propositions:

(41) Let S_1, S_2 be non void non empty many sorted signatures, X be a non-empty many sorted set indexed by the carrier of S_2, f be a function from the carrier of S_1 into the carrier of S_2, and g be a function. Suppose f and g form morphism between S_1 and S_2. Let o be an operation symbol of S_1, p be an element of Args$(o, \text{Free}(X \cdot f))$, and q be a finite sequence. Suppose $q = \text{hom}(f, g, X, S_1, S_2)\#p$. Then (hom$(f, g, X, S_1, S_2))$(the result sort of o)($\langle o$, the carrier of $S_1\rangle$-tree$(p)) = \langle g(o)$, the carrier of $S_2\rangle$-tree(q).

(42) Let S_1, S_2 be non void non empty many sorted signatures, X be a non-empty many sorted set indexed by the carrier of S_2, f be a function from the carrier of S_1 into the carrier of S_2, and g be a function. Suppose f and g form morphism between S_1 and S_2. Let t be a term of S_1 over $X \cdot f$. Then (hom$(f, g, X, S_1, S_2))$(the sort of $t)(t)$ is a compound term of S_2 over X if and only if t is a compound term of S_1 over $X \cdot f$.

(43) Let S_1, S_2 be non void non empty many sorted signatures, X be a non-empty many sorted set indexed by the carrier of S_2, f be a function from the carrier of S_1 into the carrier of S_2, and g be an one-to-

one function. Suppose f and g form morphism between S_1 and S_2. Then $\hom(f, g, X, S_1, S_2)$ is a monomorphism of $\mathrm{Free}(X \cdot f)$ into $\mathrm{Free}(X){\restriction}_{(f,g)}S_1$.

(44) Let S be a non void non empty many sorted signature and X be a non-empty many sorted set indexed by the carrier of S. Then $\hom(\mathrm{id}_{\text{the carrier of } S}, \mathrm{id}_{\text{the operation symbols of } S}, X, S, S) = \mathrm{id}_{\text{the sorts of } \mathrm{Free}(X)}$.

(45) Let S_1, S_2, S_3 be non void non empty many sorted signatures, X be a non-empty many sorted set indexed by the carrier of S_3, f_1 be a function from the carrier of S_1 into the carrier of S_2, and g_1 be a function. Suppose f_1 and g_1 form morphism between S_1 and S_2. Let f_2 be a function from the carrier of S_2 into the carrier of S_3 and g_2 be a function. Suppose f_2 and g_2 form morphism between S_2 and S_3. Then $\hom(f_2 \cdot f_1, g_2 \cdot g_1, X, S_1, S_3) = (\hom(f_2, g_2, X, S_2, S_3) \cdot f_1) \circ \hom(f_1, g_1, X \cdot f_2, S_1, S_2)$.

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Baire Spaces, Sober Spaces[1]

Andrzej Trybulec
Warsaw University
Białystok

Summary. In the article concepts and facts necessary to continue formalization of theory of continuous lattices according to [10] are introduced.

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The notation and terminology used here are introduced in the following papers: [17], [22], [21], [23], [7], [13], [2], [1], [3], [5], [9], [19], [16], [14], [24], [11], [12], [15], [6], [18], [20], [8], and [4].

1. Preliminaries

One can prove the following propositions:

(1) For all sets X, A, B such that $A \in \operatorname{Fin} X$ and $B \subseteq A$ holds $B \in \operatorname{Fin} X$.

(2) For every set X and for every family F of subsets of X such that $F \subseteq \operatorname{Fin} X$ holds $\bigcap F \in \operatorname{Fin} X$.

Let X be a non empty set. Let us observe that X is trivial if and only if:

(Def. 1) For all elements x, y of X holds $x = y$.

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2. Families of complements

We now state a number of propositions:

(3) For every set X and for every family F of subsets of X and for every subset P of X holds $P^{\mathrm{c}} \in F^{\mathrm{c}}$ iff $P \in F$.

(4) For every set X and for every family F of subsets of X holds $F \approx F^{\mathrm{c}}$.

(5) For all sets X, Y such that $X \approx Y$ and X is countable holds Y is countable.

(6) For every set X and for every family F of subsets of X holds $(F^{\mathrm{c}})^{\mathrm{c}} = F$.

(7) For every set X and for every family F of subsets of X and for every subset P of X holds $P^{\mathrm{c}} \in F^{\mathrm{c}}$ iff $P \in F$.

(8) For every set X and for all families F, G of subsets of X such that $F^{\mathrm{c}} \subseteq G^{\mathrm{c}}$ holds $F \subseteq G$.

(9) For every set X and for all families F, G of subsets of X holds $F^{\mathrm{c}} \subseteq G$ iff $F \subseteq G^{\mathrm{c}}$.

(10) For every set X and for all families F, G of subsets of X such that $F^{\mathrm{c}} = G^{\mathrm{c}}$ holds $F = G$.

(11) For every set X and for all families F, G of subsets of X holds $(F \cup G)^{\mathrm{c}} = F^{\mathrm{c}} \cup G^{\mathrm{c}}$.

(12) For every set X and for every family F of subsets of X such that $F = \{X\}$ holds $F^{\mathrm{c}} = \{\emptyset\}$.

Let X be a set and let F be an empty family of subsets of X. Observe that F^{c} is empty.

The following propositions are true:

(13) Let X be a 1-sorted structure, F be a family of subsets of X, and P be a subset of the carrier of X. Then $P \in F^{\mathrm{c}}$ if and only if $-P \in F$.

(14) Let X be a 1-sorted structure, F be a family of subsets of X, and P be a subset of the carrier of X. Then $-P \in F^{\mathrm{c}}$ if and only if $P \in F$.

(15) For every 1-sorted structure X and for every family F of subsets of X holds $\mathrm{Intersect}(F^{\mathrm{c}}) = -\bigcup F$.

(16) For every 1-sorted structure X and for every family F of subsets of X holds $\bigcup(F^{\mathrm{c}}) = -\mathrm{Intersect}(F)$.

3. Topological preliminaries

One can prove the following four propositions:

(17) Let T be a non empty topological space and A, B be subsets of the carrier of T. Suppose $B \subseteq A$ and A is closed and for every subset C of the carrier of T such that $B \subseteq C$ and C is closed holds $A \subseteq C$. Then $A = \overline{B}$.

(18) Let T be a topological structure, B be a basis of T, and V be a subset of T. If V is open, then $V = \bigcup\{G, G$ ranges over subsets of T: $G \in B \wedge G \subseteq V\}$.

(19) Let T be a topological structure, B be a basis of T, and S be a subset of T. If $S \in B$, then S is open.

(20) Let T be a non empty topological space, B be a basis of T, and V be a subset of T. Then $\operatorname{Int} V = \bigcup\{G, G$ ranges over subsets of T: $G \in B \wedge G \subseteq V\}$.

4. Baire Spaces

Let T be a non empty topological structure and let x be a point of T. A family of subsets of T is called a basis of x if it satisfies the conditions (Def. 2).

(Def. 2)(i) It $\subseteq$ the topology of T,

(ii) $x \in$ Intersect(it), and

(iii) for every subset S of T such that S is open and $x \in S$ there exists a subset V of T such that $V \in$ it and $V \subseteq S$.

Next we state three propositions:

(21) Let T be a non empty topological structure, x be a point of T, B be a basis of x, and V be a subset of T. If $V \in B$, then V is open and $x \in V$.

(22) Let T be a non empty topological structure, x be a point of T, B be a basis of x, and V be a subset of the carrier of T. If $x \in V$ and V is open, then there exists a subset W of T such that $W \in B$ and $W \subseteq V$.

(23) Let T be a non empty topological structure and P be a family of subsets of T. Suppose $P \subseteq$ the topology of T and for every point x of T there exists a basis B of x such that $B \subseteq P$. Then P is a basis of T.

Let T be a non empty topological space. We say that T is Baire if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let F be a family of subsets of T. Suppose F is countable and for every subset S of T such that $S \in F$ holds S is everywhere dense. Then Intersect(F) is dense.

We now state the proposition

(24) Let T be a non empty topological space. Then T is Baire if and only if for every family F of subsets of T such that F is countable and for every subset S of T such that $S \in F$ holds S is nowhere dense holds $\bigcup F$ is boundary.

5. Sober Spaces

Let T be a topological structure and let S be a subset of T. We say that S is irreducible if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) S is non empty and closed, and

(ii) for all subsets S_1, S_2 of T such that S_1 is closed and S_2 is closed and $S = S_1 \cup S_2$ holds $S_1 = S$ or $S_2 = S$.

Let T be a topological structure. Observe that every subset of T which is irreducible is also non empty.

Let T be a non empty topological space, let S be a subset of the carrier of T, and let p be a point of T. We say that p is dense point of S if and only if:

(Def. 5) $p \in S$ and $S \subseteq \overline{\{p\}}$.

We now state two propositions:

(25) Let T be a non empty topological space and S be a subset of the carrier of T. Suppose S is closed. Let p be a point of T. If p is dense point of S, then $S = \overline{\{p\}}$.

(26) For every non empty topological space T and for every point p of T holds $\overline{\{p\}}$ is irreducible.

Let T be a non empty topological space. Observe that there exists a subset of T which is irreducible.

Let T be a non empty topological space. We say that T is sober if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let S be an irreducible subset of T. Then there exists a point p of T such that p is dense point of S and for every point q of T such that q is dense point of S holds $p = q$.

We now state four propositions:

(27) For every non empty topological space T and for every point p of T holds p is dense point of $\overline{\{p\}}$.

(28) For every non empty topological space T and for every point p of T holds p is dense point of $\{p\}$.

(29) Let T be a non empty topological space and G, F be subsets of T. If G is open and F is closed, then $F \setminus G$ is closed.

(30) For every Hausdorff non empty topological space T holds every irreducible subset of T is trivial.

Let T be a Hausdorff non empty topological space. Observe that every subset of T which is irreducible is also trivial.

We now state the proposition

(31) Every Hausdorff non empty topological space is sober.

Let us note that every non empty topological space which is Hausdorff is also sober.

One can verify that there exists a non empty topological space which is sober.

The following two propositions are true:

(32) Let T be a non empty topological space. Then T is T_0 if and only if for all points p, q of T such that $\overline{\{p\}} = \overline{\{q\}}$ holds $p = q$.

(33) Every sober non empty topological space is T_0.

Let us note that every non empty topological space which is sober is also T_0.

Let X be a set. The functor $\mathrm{CofinTop}\,X$ yields a strict topological structure and is defined as follows:

(Def. 7) The carrier of $\mathrm{CofinTop}\,X = X$ and (the topology of $\mathrm{CofinTop}\,X)^{\mathrm{c}} = \{X\} \cup \mathrm{Fin}\,X$.

Let X be a non empty set. Note that $\mathrm{CofinTop}\,X$ is non empty.

Let X be a set. Note that $\mathrm{CofinTop}\,X$ is topological space-like.

Next we state two propositions:

(34) For every non empty set X and for every subset P of $\mathrm{CofinTop}\,X$ holds P is closed iff $P = X$ or P is finite.

(35) For every non empty topological space T such that T is a T_1 space and for every point p of T holds $\overline{\{p\}} = \{p\}$.

Let X be a non empty set. Note that $\mathrm{CofinTop}\,X$ is a T_1 space.

Let X be an infinite set. One can check that $\mathrm{CofinTop}\,X$ is non sober.

Let us observe that there exists a non empty topological space which is a T_1 space and non sober.

6. More on regular spaces

One can prove the following two propositions:

(36) Let T be a non empty topological space. Then T is a T_3 space if and only if for every point p of T and for every subset P of the carrier of T such that $p \in \mathrm{Int}\,P$ there exists a subset Q of T such that Q is closed and $Q \subseteq P$ and $p \in \mathrm{Int}\,Q$.

(37) Let T be a non empty topological space. Suppose T is a T_3 space. Then T is locally-compact if and only if for every point x of T there exists a subset Y of T such that $x \in \mathrm{Int}\,Y$ and Y is compact.

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Closure Operators and Subalgebras[1]

Grzegorz Bancerek
Warsaw University
Białystok

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The notation and terminology used in this paper are introduced in the following papers: [19], [22], [11], [23], [24], [9], [10], [1], [4], [18], [15], [17], [20], [2], [21], [3], [16], [13], [5], [6], [14], [25], [12], [8], and [7].

1. Preliminaries

In this article we present several logical schemes. The scheme *SubrelstrEx* concerns a non empty relational structure $\mathcal{A}$, a set $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

> There exists a non empty full strict relational substructure S of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds x is an element of S if and only if $\mathcal{P}[x]$

provided the following conditions are met:

- $\mathcal{P}[\mathcal{B}]$,
- $\mathcal{B} \in$ the carrier of $\mathcal{A}$.

The scheme *RelstrEq* deals with non empty relational structures $\mathcal{A}$, $\mathcal{B}$, a unary predicate $\mathcal{P}$, and a binary predicate $\mathcal{Q}$, and states that:

> The relational structure of $\mathcal{A}$ = the relational structure of $\mathcal{B}$

provided the following conditions are met:

- For every set x holds x is an element of $\mathcal{A}$ iff $\mathcal{P}[x]$,
- For every set x holds x is an element of $\mathcal{B}$ iff $\mathcal{P}[x]$,
- For all elements a, b of $\mathcal{A}$ holds $a \leqslant b$ iff $\mathcal{Q}[a, b]$,
- For all elements a, b of $\mathcal{B}$ holds $a \leqslant b$ iff $\mathcal{Q}[a, b]$.

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The scheme *SubrelstrEq1* deals with a non empty relational structure $\mathcal{A}$, non empty full relational substructures $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

The relational structure of $\mathcal{B}$ = the relational structure of $\mathcal{C}$

provided the following conditions are satisfied:

- For every set x holds x is an element of $\mathcal{B}$ iff $\mathcal{P}[x]$,
- For every set x holds x is an element of $\mathcal{C}$ iff $\mathcal{P}[x]$.

The scheme *SubrelstrEq2* concerns a non empty relational structure $\mathcal{A}$, non empty full relational substructures $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

The relational structure of $\mathcal{B}$ = the relational structure of $\mathcal{C}$

provided the parameters have the following properties:

- For every element x of $\mathcal{A}$ holds x is an element of $\mathcal{B}$ iff $\mathcal{P}[x]$,
- For every element x of $\mathcal{A}$ holds x is an element of $\mathcal{C}$ iff $\mathcal{P}[x]$.

The following four propositions are true:

(1) For all binary relations R, Q holds $R \subseteq Q$ iff $R^{\smile} \subseteq Q^{\smile}$ and $R^{\smile} \subseteq Q$ iff $R \subseteq Q^{\smile}$.

(2) For every binary relation R and for every set X holds $(R|^2X)^{\smile} = R^{\smile}|^2X$.

(3) Let L, S be relational structures. Then

(i) S is a relational substructure of L iff $S^{\rm op}$ is a relational substructure of $L^{\rm op}$, and

(ii) $S^{\rm op}$ is a relational substructure of L iff S is a relational substructure of $L^{\rm op}$.

(4) Let L, S be relational structures. Then

(i) S is a full relational substructure of L iff $S^{\rm op}$ is a full relational substructure of $L^{\rm op}$, and

(ii) $S^{\rm op}$ is a full relational substructure of L iff S is a full relational substructure of $L^{\rm op}$.

Let L be a relational structure and let S be a full relational substructure of L. Then $S^{\rm op}$ is a strict full relational substructure of $L^{\rm op}$.

Let X be a set and let L be a non empty relational structure. Observe that $X \longmapsto L$ is nonempty.

Let S be a relational structure and let T be a non empty reflexive relational structure. One can verify that there exists a map from S into T which is monotone.

Let L be a non empty relational structure. One can check that every map from L into L which is projection is also monotone and idempotent.

Let S, T be non empty reflexive relational structures and let f be a monotone map from S into T. One can verify that f° is monotone.

Let L be a 1-sorted structure. Note that id_L is one-to-one.

Let L be a non empty reflexive relational structure. One can check that id_L is sups-preserving and infs-preserving.

The following proposition is true

(5) Let L be a relational structure and S be a subset of L. Then id_S is a map from $\mathrm{sub}(S)$ into L and for every map f from $\mathrm{sub}(S)$ into L such that $f = \mathrm{id}_S$ holds f is monotone.

Let L be a non empty reflexive relational structure. Note that there exists a map from L into L which is sups-preserving, infs-preserving, closure, kernel, and one-to-one.

One can prove the following proposition

(6) Let L be a non empty reflexive relational structure, c be a closure map from L into L, and x be an element of L. Then $c(x) \geqslant x$.

Let S, T be 1-sorted structures, let f be a function from the carrier of S into the carrier of T, and let R be a 1-sorted structure. Let us assume that the carrier of $R \subseteq$ the carrier of S. The functor $f{\restriction}R$ yields a map from R into T and is defined by:

(Def. 1) $f{\restriction}R = f{\restriction}$the carrier of R.

One can prove the following propositions:

(7) Let S, T be relational structures, R be a relational substructure of S, and f be a function from the carrier of S into the carrier of T. Then $f{\restriction}R = f{\restriction}$the carrier of R and for every set x such that $x \in$ the carrier of R holds $(f{\restriction}R)(x) = f(x)$.

(8) Let S, T be relational structures and f be a map from S into T. Suppose f is one-to-one. Let R be a relational substructure of S. Then $f{\restriction}R$ is one-to-one.

Let S, T be non empty reflexive relational structures, let f be a monotone map from S into T, and let R be a relational substructure of S. Note that $f{\restriction}R$ is monotone.

One can prove the following proposition

(9) Let S, T be non empty relational structures, R be a non empty relational substructure of S, f be a map from S into T, and g be a map from T into S. Suppose f is one-to-one and $g = f^{-1}$. Then $g{\restriction}\mathrm{Im}(f{\restriction}R)$ is a map from $\mathrm{Im}(f{\restriction}R)$ into R and $g{\restriction}\mathrm{Im}(f{\restriction}R) = (f{\restriction}R)^{-1}$.

2. The lattice of closure operators

Let S be a relational structure and let T be a non empty reflexive relational structure. Note that $\mathrm{MonMaps}(S, T)$ is non empty.

Next we state the proposition

(10) Let S be a relational structure, T be a non empty reflexive relational structure, and x be a set. Then x is an element of $\mathrm{MonMaps}(S, T)$ if and only if x is a monotone map from S into T.

Let L be a non empty reflexive relational structure. The functor $\mathrm{ClOpers}(L)$ yields a non empty full strict relational substructure of $\mathrm{MonMaps}(L, L)$ and is defined by:

(Def. 2) For every map f from L into L holds f is an element of $\mathrm{ClOpers}(L)$ iff f is closure.

The following propositions are true:

(11) Let L be a non empty reflexive relational structure and x be a set. Then x is an element of $\mathrm{ClOpers}(L)$ if and only if x is a closure map from L into L.

(12) Let X be a set, L be a non empty relational structure, f, g be functions from X into the carrier of L, and x, y be elements of L^X. If $x = f$ and $y = g$, then $x \leqslant y$ iff $f \leqslant g$.

(13) Let L be a complete lattice, c_1, c_2 be maps from L into L, and x, y be elements of $\mathrm{ClOpers}(L)$. If $x = c_1$ and $y = c_2$, then $x \leqslant y$ iff $c_1 \leqslant c_2$.

(14) Let L be a reflexive relational structure and S_1, S_2 be full relational substructures of L. Suppose the carrier of $S_1 \subseteq$ the carrier of S_2. Then S_1 is a relational substructure of S_2.

(15) Let L be a complete lattice and c_1, c_2 be closure maps from L into L. Then $c_1 \leqslant c_2$ if and only if $\mathrm{Im}\, c_2$ is a relational substructure of $\mathrm{Im}\, c_1$.

3. The lattice of closure systems

Let L be a relational structure. The functor $\mathrm{Sub}(L)$ yields a strict non empty relational structure and is defined by the conditions (Def. 3).

(Def. 3)(i) For every set x holds x is an element of $\mathrm{Sub}(L)$ iff x is a strict relational substructure of L, and

(ii) for all elements a, b of $\mathrm{Sub}(L)$ holds $a \leqslant b$ iff there exists a relational structure R such that $b = R$ and a is a relational substructure of R.

One can prove the following proposition

(16) Let L, R be relational structures and x, y be elements of $\mathrm{Sub}(L)$. Suppose $y = R$. Then $x \leqslant y$ if and only if x is a relational substructure of R.

Let L be a relational structure. One can verify that $\mathrm{Sub}(L)$ is reflexive antisymmetric and transitive.

Let L be a relational structure. Observe that $\mathrm{Sub}(L)$ is complete.

Let L be a complete lattice. Note that every relational substructure of L which is infs-inheriting is also non empty and every relational substructure of L which is sups-inheriting is also non empty.

Let L be a relational structure. A system of L is a full relational substructure of L.

Let L be a non empty relational structure and let S be a system of L. We introduce S is closure as a synonym of S is infs-inheriting.

Let L be a non empty relational structure. Observe that $\mathrm{sub}(\Omega_L)$ is infs-inheriting and sups-inheriting.

Let L be a non empty relational structure. The functor ClosureSystems(L) yields a full strict non empty relational substructure of Sub(L) and is defined by the condition (Def. 4).

(Def. 4) Let R be a strict relational substructure of L. Then R is an element of ClosureSystems(L) if and only if R is infs-inheriting and full.

Next we state two propositions:

(17) Let L be a non empty relational structure and x be a set. Then x is an element of ClosureSystems(L) if and only if x is a strict closure system of L.

(18) Let L be a non empty relational structure, R be a relational structure, and x, y be elements of ClosureSystems(L). Suppose $y = R$. Then $x \leqslant y$ if and only if x is a relational substructure of R.

4. Isomorphism between closure operators and closure systems

Let L be a non empty poset and let h be a closure map from L into L. Note that $\mathrm{Im}\,h$ is infs-inheriting.

Let L be a non empty poset. The functor ClImageMap(L) yields a map from ClOpers(L) into $(\mathrm{ClosureSystems}(L))^{\mathrm{op}}$ and is defined as follows:

(Def. 5) For every closure map c from L into L holds $(\mathrm{ClImageMap}(L))(c) = \mathrm{Im}\,c$.

Let L be a non empty relational structure and let S be a relational substructure of L. The closure operation of S is a map from L into L and is defined by:

(Def. 6) For every element x of L holds (the closure operation of S)(x) $= \bigsqcap_L(\uparrow x \cap$ the carrier of S).

Let L be a complete lattice and let S be a closure system of L. One can verify that the closure operation of S is closure.

Next we state two propositions:

(19) Let L be a complete lattice and S be a closure system of L. Then Im (the closure operation of S) = the relational structure of S.

(20) For every complete lattice L and for every closure map c from L into L holds the closure operation of $\mathrm{Im}\,c = c$.

Let L be a complete lattice. One can check that ClImageMap(L) is one-to-one.

One can prove the following propositions:

(21) For every complete lattice L holds $(\mathrm{ClImageMap}(L))^{-1}$ is a map from $(\mathrm{ClosureSystems}(L))^{\mathrm{op}}$ into ClOpers(L).

(22) Let L be a complete lattice and S be a strict closure system of L. Then $(\mathrm{ClImageMap}(L))^{-1}(S)$ = the closure operation of S.

Let L be a complete lattice. One can verify that ClImageMap(L) is isomorphic.

The following proposition is true

(23) For every complete lattice L holds $\mathrm{ClOpers}(L)$ and $(\mathrm{ClosureSystems}(L))^{\mathrm{op}}$ are isomorphic.

5. Isomorphism between closure operators preserving directed sups and subalgebras

We now state three propositions:

(24) Let L be a relational structure, S be a full relational substructure of L, and X be a subset of S. Then
 (i) if X is a directed subset of L, then X is directed, and
 (ii) if X is a filtered subset of L, then X is filtered.

(25) Let L be a complete lattice and S be a closure system of L. Then the closure operation of S is directed-sups-preserving if and only if S is directed-sups-inheriting.

(26) Let L be a complete lattice and h be a closure map from L into L. Then h is directed-sups-preserving if and only if $\mathrm{Im}\, h$ is directed-sups-inheriting.

Let L be a complete lattice and let S be a directed-sups-inheriting closure system of L. Observe that the closure operation of S is directed-sups-preserving.

Let L be a complete lattice and let h be a directed-sups-preserving closure map from L into L. Observe that $\mathrm{Im}\, h$ is directed-sups-inheriting.

Let L be a non empty reflexive relational structure. The functor $\mathrm{ClOpers}^*(L)$ yields a non empty full strict relational substructure of $\mathrm{ClOpers}(L)$ and is defined by the condition (Def. 7).

(Def. 7) Let f be a closure map from L into L. Then f is an element of $\mathrm{ClOpers}^*(L)$ if and only if f is directed-sups-preserving.

Next we state the proposition

(27) Let L be a non empty reflexive relational structure and x be a set. Then x is an element of $\mathrm{ClOpers}^*(L)$ if and only if x is a directed-sups-preserving closure map from L into L.

Let L be a non empty relational structure. The functor $\mathrm{Subalgebras}(L)$ yields a full strict non empty relational substructure of $\mathrm{ClosureSystems}(L)$ and is defined by the condition (Def. 8).

(Def. 8) Let R be a strict closure system of L. Then R is an element of $\mathrm{Subalgebras}(L)$ if and only if R is directed-sups-inheriting.

The following two propositions are true:

(28) Let L be a non empty relational structure and x be a set. Then x is an element of $\mathrm{Subalgebras}(L)$ if and only if x is a strict directed-sups-inheriting closure system of L.

(29) For every complete lattice L holds $\mathrm{Im}(\mathrm{ClImageMap}(L){\restriction}\mathrm{ClOpers}^*(L)) = (\mathrm{Subalgebras}(L))^{\mathrm{op}}$.

Let L be a complete lattice. Note that $(\mathrm{ClImageMap}(L) \restriction \mathrm{ClOpers}^*(L))^\circ$ is isomorphic.

The following proposition is true

(30) For every complete lattice L holds $\mathrm{ClOpers}^*(L)$ and $(\mathrm{Subalgebras}(L))^{\mathrm{op}}$ are isomorphic.

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Algebra of Morphisms

Grzegorz Bancerek
Warsaw University
Białystok

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The papers [22], [27], [15], [2], [23], [21], [28], [11], [12], [14], [9], [10], [19], [3], [26], [1], [4], [24], [18], [25], [17], [20], [6], [16], [5], [7], [8], and [13] provide the notation and terminology for this paper.

1. Preliminaries

Let I be a set and let A, f be functions. The functor $f\restriction_I A$ yielding a many sorted function indexed by I is defined by:

(Def. 1) For every set i such that $i \in I$ holds $(f\restriction_I A)(i) = f\restriction A(i)$.

One can prove the following propositions:

(1) For every set I and for every many sorted set A indexed by I holds $\mathrm{id}_{\mathrm{Union}\,A}\restriction_I A = \mathrm{id}_A$.

(2) Let I be a set, A, B be many sorted sets indexed by I, and f, g be functions. If $\mathrm{rng}_\kappa(f\restriction_I A)(\kappa) \subseteq B$, then $(g \cdot f)\restriction_I A = (g\restriction_I B) \circ (f\restriction_I A)$.

(3) Let f be a function, I be a set, and A, B be many sorted sets indexed by I. Suppose that for every set i such that $i \in I$ holds $A(i) \subseteq \mathrm{dom}\, f$ and $f^\circ A(i) \subseteq B(i)$. Then $f\restriction_I A$ is a many sorted function from A into B.

(4) Let A be a set, i be a natural number, and p be a finite sequence. Then $p \in A^i$ if and only if $\mathrm{len}\, p = i$ and $\mathrm{rng}\, p \subseteq A$.

(5) Let A be a set, i be a natural number, and p be a finite sequence of elements of A. Then $p \in A^i$ if and only if $\mathrm{len}\, p = i$.

(6) For every set A and for every natural number i holds $A^i \subseteq A^*$.

(7) For every set A and for every natural number i holds $i \neq 0$ and $A = \emptyset$ iff $A^i = \emptyset$.

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(8) For all sets A, x holds $x \in A^1$ iff there exists a set a such that $a \in A$ and $x = \langle a \rangle$.

(9) For all sets A, a such that $\langle a \rangle \in A^1$ holds $a \in A$.

(10) For all sets A, x holds $x \in A^2$ iff there exist sets a, b such that $a \in A$ and $b \in A$ and $x = \langle a, b \rangle$.

(11) For all sets A, a, b such that $\langle a, b \rangle \in A^2$ holds $a \in A$ and $b \in A$.

(12) For all sets A, x holds $x \in A^3$ iff there exist sets a, b, c such that $a \in A$ and $b \in A$ and $c \in A$ and $x = \langle a, b, c \rangle$.

(13) For all sets A, a, b, c such that $\langle a, b, c \rangle \in A^3$ holds $a \in A$ and $b \in A$ and $c \in A$.

Let A be a function. We say that A is mutually-disjoint if and only if:

(Def. 2) For all sets x, y such that $x \neq y$ holds $A(x)$ misses $A(y)$.

Let S be a non empty many sorted signature and let A be an algebra over S. We say that A is empty if and only if:

(Def. 3) The sorts of A are empty yielding.

We say that A is disjoint if and only if:

(Def. 4) The sorts of A are mutually-disjoint.

Let S be a non empty many sorted signature. Note that every algebra over S which is non-empty is also non empty.

Let S be a non empty non void many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. One can check that $\mathrm{Free}(X)$ is disjoint.

Let S be a non empty non void many sorted signature. Observe that there exists an algebra over S which is strict, non-empty, and disjoint.

Let S be a non empty non void many sorted signature and let A be a non empty algebra over S. One can verify that the sorts of A is non empty yielding.

One can verify that there exists a function which is non empty yielding.

2. Signature of a category

Let A be a set. The functor $\mathrm{CatSign}(A)$ yielding a strict many sorted signature is defined by the conditions (Def. 5).

(Def. 5)(i) The carrier of $\mathrm{CatSign}(A) = [\!: \{0\}, A^2 :\!]$,

(ii) the operation symbols of $\mathrm{CatSign}(A) = [\!: \{1\}, A^1 :\!] \cup [\!: \{2\}, A^3 :\!]$,

(iii) for every set a such that $a \in A$ holds (the arity of $\mathrm{CatSign}(A)$)($\langle 1, \langle a \rangle \rangle$) $= \varepsilon$ and (the result sort of $\mathrm{CatSign}(A)$)($\langle 1, \langle a \rangle \rangle$) $= \langle 0, \langle a, a \rangle \rangle$, and

(iv) for all sets a, b, c such that $a \in A$ and $b \in A$ and $c \in A$ holds (the arity of $\mathrm{CatSign}(A)$)($\langle 2, \langle a, b, c \rangle \rangle$) $= \langle \langle 0, \langle b, c \rangle \rangle, \langle 0, \langle a, b \rangle \rangle \rangle$ and (the result sort of $\mathrm{CatSign}(A)$)($\langle 2, \langle a, b, c \rangle \rangle$) $= \langle 0, \langle a, c \rangle \rangle$.

Let A be a set. Observe that $\mathrm{CatSign}(A)$ is feasible.

Let A be a non empty set. Observe that $\mathrm{CatSign}(A)$ is non empty and non void.

Instead of a feasible many sorted signature we will use a signature.

Let S be a signature. We say that S is categorial if and only if:

(Def. 6) There exists a set A such that $\mathrm{CatSign}(A)$ is a subsignature of S and the carrier of $S = [: \{0\}, A^2 :]$.

Let us note that every non empty signature which is categorial is also non void.

One can check that there exists a signature which is categorial, non empty, and strict.

A cat-signature is a categorial signature.

Let A be a set. A signature is said to be a cat-signature of A if:

(Def. 7) $\mathrm{CatSign}(A)$ is a subsignature of it and the carrier of it $= [: \{0\}, A^2 :]$.

One can prove the following proposition

(14) For all sets A_1, A_2 and for every cat-signature S of A_1 such that S is a cat-signature of A_2 holds $A_1 = A_2$.

Let A be a set. Note that every cat-signature of A is categorial.

Let A be a non empty set. Note that every cat-signature of A is non empty.

Let A be a set. Observe that there exists a cat-signature of A which is strict.

Let A be a set. Then $\mathrm{CatSign}(A)$ is a strict cat-signature of A.

Let S be a many sorted signature. The functor $\mathrm{underlay}\, S$ is defined by the condition (Def. 8).

(Def. 8) Let x be a set. Then $x \in \mathrm{underlay}\, S$ if and only if there exists a set a and there exists a function f such that $\langle a, f \rangle \in (\text{the carrier of } S) \cup (\text{the operation symbols of } S)$ and $x \in \mathrm{rng}\, f$.

One can prove the following proposition

(15) For every set A holds $\mathrm{underlay}\, \mathrm{CatSign}(A) = A$.

Let S be a many sorted signature. We say that S is δ-concrete if and only if the condition (Def. 9) is satisfied.

(Def. 9) There exists a function f from $\mathbb{N}$ into $\mathbb{N}$ such that
- (i) for every set s such that $s \in$ the carrier of S there exists a natural number i and there exists a finite sequence p such that $s = \langle i, p \rangle$ and $\mathrm{len}\, p = f(i)$ and $[: \{i\}, (\mathrm{underlay}\, S)^{f(i)} :] \subseteq$ the carrier of S, and
- (ii) for every set o such that $o \in$ the operation symbols of S there exists a natural number i and there exists a finite sequence p such that $o = \langle i, p \rangle$ and $\mathrm{len}\, p = f(i)$ and $[: \{i\}, (\mathrm{underlay}\, S)^{f(i)} :] \subseteq$ the operation symbols of S.

Let A be a set. One can check that $\mathrm{CatSign}(A)$ is δ-concrete.

Observe that there exists a cat-signature which is δ-concrete, non empty, and strict. Let A be a set. One can check that there exists a cat-signature of A which is δ-concrete and strict.

The following propositions are true:

(16) Let S be a δ-concrete many sorted signature and x be a set. Suppose $x \in$ the carrier of S or $x \in$ the operation symbols of S. Then there exists a natural number i and there exists a finite sequence p such that $x = \langle i, p \rangle$ and $\operatorname{rng} p \subseteq \operatorname{underlay} S$.

(17) Let S be a δ-concrete many sorted signature, i be a set, and p_1, p_2 be finite sequences. Suppose that

(i) $\langle i, p_1 \rangle \in$ the carrier of S and $\langle i, p_2 \rangle \in$ the carrier of S, or

(ii) $\langle i, p_1 \rangle \in$ the operation symbols of S and $\langle i, p_2 \rangle \in$ the operation symbols of S.

Then $\operatorname{len} p_1 = \operatorname{len} p_2$.

(18) Let S be a δ-concrete many sorted signature, i be a set, and p_1, p_2 be finite sequences such that $\operatorname{len} p_2 = \operatorname{len} p_1$ and $\operatorname{rng} p_2 \subseteq \operatorname{underlay} S$. Then

(i) if $\langle i, p_1 \rangle \in$ the carrier of S, then $\langle i, p_2 \rangle \in$ the carrier of S, and

(ii) if $\langle i, p_1 \rangle \in$ the operation symbols of S, then $\langle i, p_2 \rangle \in$ the operation symbols of S.

(19) Every δ-concrete categorial non empty signature S is a cat-signature of underlay S.

3. Symbols of categorial signatures

Let S be a non empty cat-signature and let s be a sort symbol of S. Note that $s_{\mathbf{2}}$ is relation-like and function-like.

Let S be a non empty δ-concrete many sorted signature and let s be a sort symbol of S. Observe that $s_{\mathbf{2}}$ is relation-like and function-like.

Let S be a non void δ-concrete many sorted signature and let o be an element of the operation symbols of S. One can verify that $o_{\mathbf{2}}$ is relation-like and function-like.

Let S be a non empty cat-signature and let s be a sort symbol of S. One can verify that $s_{\mathbf{2}}$ is finite sequence-like.

Let S be a non empty δ-concrete many sorted signature and let s be a sort symbol of S. Observe that $s_{\mathbf{2}}$ is finite sequence-like.

Let S be a non void δ-concrete many sorted signature and let o be an element of the operation symbols of S. Observe that $o_{\mathbf{2}}$ is finite sequence-like.

Let a be a set. The functor $\operatorname{idsym} a$ is defined as follows:

(Def. 10) $\operatorname{idsym} a = \langle 1, \langle a \rangle \rangle$.

Let b be a set. The functor $\operatorname{homsym}(a, b)$ is defined as follows:

(Def. 11) $\operatorname{homsym}(a, b) = \langle 0, \langle a, b \rangle \rangle$.

Let c be a set. The functor $\operatorname{compsym}(a, b, c)$ is defined as follows:

(Def. 12) $\operatorname{compsym}(a, b, c) = \langle 2, \langle a, b, c \rangle \rangle$.

Next we state the proposition

(20) Let A be a non empty set, S be a cat-signature of A, and a be an element of A. Then

(i) $\mathrm{idsym}\, a \in$ the operation symbols of S, and

(ii) for every element b of A holds $\mathrm{homsym}(a, b) \in$ the carrier of S and for every element c of A holds $\mathrm{compsym}(a, b, c) \in$ the operation symbols of S.

Let A be a non empty set and let a be an element of A. Then $\mathrm{idsym}\, a$ is an operation symbol of $\mathrm{CatSign}(A)$. Let b be an element of A. Then $\mathrm{homsym}(a, b)$ is a sort symbol of $\mathrm{CatSign}(A)$. Let c be an element of A. Then $\mathrm{compsym}(a, b, c)$ is an operation symbol of $\mathrm{CatSign}(A)$.

We now state several propositions:

(21) For all sets a, b such that $\mathrm{idsym}\, a = \mathrm{idsym}\, b$ holds $a = b$.

(22) For all sets a_1, b_1, a_2, b_2 such that $\mathrm{homsym}(a_1, a_2) = \mathrm{homsym}(b_1, b_2)$ holds $a_1 = b_1$ and $a_2 = b_2$.

(23) For all sets a_1, b_1, a_2, b_2, a_3, b_3 such that $\mathrm{compsym}(a_1, a_2, a_3) = \mathrm{compsym}(b_1, b_2, b_3)$ holds $a_1 = b_1$ and $a_2 = b_2$ and $a_3 = b_3$.

(24) Let A be a non empty set, S be a cat-signature of A and s be a sort symbol of S. Then there exist elements a, b of A such that $s = \mathrm{homsym}(a, b)$.

(25) For every non empty set A and for every operation symbol o of $\mathrm{CatSign}(A)$ holds $o_{\mathbf{1}} = 1$ and $\mathrm{len}(o_{\mathbf{2}}) = 1$ or $o_{\mathbf{1}} = 2$ and $\mathrm{len}(o_{\mathbf{2}}) = 3$.

(26) Let A be a non empty set and o be an operation symbol of $\mathrm{CatSign}(A)$. If $o_{\mathbf{1}} = 1$ or $\mathrm{len}(o_{\mathbf{2}}) = 1$, then there exists an element a of A such that $o = \mathrm{idsym}\, a$.

(27) Let A be a non empty set and o be an operation symbol of $\mathrm{CatSign}(A)$. If $o_{\mathbf{1}} = 2$ or $\mathrm{len}(o_{\mathbf{2}}) = 3$, then there exist elements a, b, c of A such that $o = \mathrm{compsym}(a, b, c)$.

(28) For every non empty set A and for every element a of A holds $\mathrm{Arity}(\mathrm{idsym}\, a) = \varepsilon$ and the result sort of $\mathrm{idsym}\, a = \mathrm{homsym}(a, a)$.

(29) For every non empty set A and for all elements a, b, c of A holds $\mathrm{Arity}(\mathrm{compsym}(a, b, c)) = \langle \mathrm{homsym}(b, c), \mathrm{homsym}(a, b) \rangle$ and the result sort of $\mathrm{compsym}(a, b, c) = \mathrm{homsym}(a, c)$.

4. Signature homomorphism generated by a functor

Let C_1, C_2 be categories and let F be a functor from C_1 to C_2. The functor Υ_F yields a function from the carrier of CatSign(the objects of C_1) into the carrier of CatSign(the objects of C_2) and is defined as follows:

(Def. 13) For every sort symbol s of CatSign(the objects of C_1) holds $\Upsilon_F(s) = \langle 0, \mathrm{Obj}\, F \cdot s_{\mathbf{2}} \rangle$.

The functor Ψ_F yields a function from the operation symbols of CatSign(the objects of C_1) into the operation symbols of CatSign(the objects of C_2) and is defined as follows:

(Def. 14) For every operation symbol o of CatSign(the objects of C_1) holds $\Psi_F(o) = \langle o_{\mathbf{1}}, \operatorname{Obj} F \cdot o_{\mathbf{2}} \rangle$.

The following propositions are true:

(30) For all categories C_1, C_2 and for every functor F from C_1 to C_2 and for all objects a, b of C_1 holds $\Upsilon_F(\mathrm{homsym}(a,b)) = \mathrm{homsym}(F(a), F(b))$.

(31) For all categories C_1, C_2 and for every functor F from C_1 to C_2 and for every object a of C_1 holds $\Psi_F(\mathrm{idsym}\, a) = \mathrm{idsym}\, F(a)$.

(32) Let C_1, C_2 be categories, F be a functor from C_1 to C_2, and a, b, c be objects of C_1. Then $\Psi_F(\mathrm{compsym}(a,b,c)) = \mathrm{compsym}(F(a), F(b), F(c))$.

(33) Let C_1, C_2 be categories and F be a functor from C_1 to C_2. Then Υ_F and Ψ_F form morphism between CatSign(the objects of C_1) and CatSign(the objects of C_2).

5. Algebra of morphisms

Next we state the proposition

(34) For every non empty set C and for every algebra A over CatSign(C) and for every element a of C holds $\mathrm{Args}(\mathrm{idsym}\, a, A) = \{\varepsilon\}$.

The scheme *CatAlgEx* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, a 5-ary functor $\mathcal{G}$ yielding a set, and a unary functor $\mathcal{H}$ yielding a set, and states that:

There exists a strict algebra A over CatSign($\mathcal{A}$) such that

(i) for all elements a, b of $\mathcal{A}$ holds (the sorts of A)$(\mathrm{homsym}(a,b)) = \mathcal{F}(a,b)$,

(ii) for every element a of $\mathcal{A}$ holds $(\mathrm{Den}(\mathrm{idsym}\, a, A))(\varepsilon) = \mathcal{H}(a)$, and

(iii) for all elements a, b, c of $\mathcal{A}$ and for all elements f, g of $\mathcal{B}$ such that $f \in \mathcal{F}(a,b)$ and $g \in \mathcal{F}(b,c)$ holds $(\mathrm{Den}(\mathrm{compsym}(a,b,c), A))(\langle g, f \rangle) = \mathcal{G}(a,b,c,g,f)$

provided the parameters have the following properties:

- For all elements a, b of $\mathcal{A}$ holds $\mathcal{F}(a,b) \subseteq \mathcal{B}$,
- For every element a of $\mathcal{A}$ holds $\mathcal{H}(a) \in \mathcal{F}(a,a)$,
- For all elements a, b, c of $\mathcal{A}$ and for all elements f, g of $\mathcal{B}$ such that $f \in \mathcal{F}(a,b)$ and $g \in \mathcal{F}(b,c)$ holds $\mathcal{G}(a,b,c,g,f) \in \mathcal{F}(a,c)$.

Let C be a category. The functor MSAlg(C) yielding a strict algebra over CatSign(the objects of C) is defined by the conditions (Def. 15).

(Def. 15)(i) For all objects a, b of C holds (the sorts of MSAlg(C))$(\mathrm{homsym}(a,b)) = \hom(a,b)$,

(ii) for every object a of C holds $(\mathrm{Den}(\mathrm{idsym}\, a, \mathrm{MSAlg}(C)))(\varepsilon) = \mathrm{id}_a$, and

(iii) for all objects a, b, c of C and for all morphisms f, g of C such that $\operatorname{dom} f = a$ and $\operatorname{cod} f = b$ and $\operatorname{dom} g = b$ and $\operatorname{cod} g = c$ holds $(\mathrm{Den}(\mathrm{compsym}(a,b,c), \mathrm{MSAlg}(C)))(\langle g, f \rangle) = g \cdot f$.

The following propositions are true:

(35) For every category A and for all objects a, b of A holds (the sorts of $\mathrm{MSAlg}(A))(\mathrm{homsym}(a,b)) = \hom(a,b)$.

(36) For every category A and for every object a of A holds $\mathrm{Result}(\mathrm{idsym}\,a, \mathrm{MSAlg}(A)) = \hom(a,a)$.

(37) For every category A and for all objects a, b, c of A holds $\mathrm{Args}(\mathrm{compsym}(a,b,c), \mathrm{MSAlg}(A)) = \prod\langle\hom(b,c), \hom(a,b)\rangle$ and $\mathrm{Result}(\mathrm{compsym}(a,b,c), \mathrm{MSAlg}(A)) = \hom(a,c)$.

Let C be a category. Note that $\mathrm{MSAlg}(C)$ is disjoint and feasible.

One can prove the following propositions:

(38) Let C_1, C_2 be categories and F be a functor from C_1 to C_2. Then $F\restriction_{\text{the carrier of CatSign(the objects of } C_1)}$the sorts of $\mathrm{MSAlg}(C_1)$ is a many sorted function from $\mathrm{MSAlg}(C_1)$ into $\mathrm{MSAlg}(C_2)\restriction_{(\Upsilon_F,\Psi_F)}$ CatSign(the objects of C_1).

(39) Let C be a category, a, b, c be objects of C, and x be a set. Then $x \in \mathrm{Args}(\mathrm{compsym}(a,b,c), \mathrm{MSAlg}(C))$ if and only if there exist morphisms g, f of C such that $x = \langle g, f\rangle$ and $\mathrm{dom}\, f = a$ and $\mathrm{cod}\, f = b$ and $\mathrm{dom}\, g = b$ and $\mathrm{cod}\, g = c$.

(40) Let C_1, C_2 be categories, F be a functor from C_1 to C_2, a, b, c be objects of C_1, and f, g be morphisms of C_1. Suppose $f \in \hom(a,b)$ and $g \in \hom(b,c)$. Let x be an element of $\mathrm{Args}(\mathrm{compsym}(a,b,c), \mathrm{MSAlg}(C_1))$. Suppose $x = \langle g, f\rangle$. Let H be a many sorted function from $\mathrm{MSAlg}(C_1)$ into $\mathrm{MSAlg}(C_2)\restriction_{(\Upsilon_F,\Psi_F)}$ CatSign(the objects of C_1). Suppose $H = F\restriction_{\text{the carrier of CatSign(the objects of } C_1)}$the sorts of $\mathrm{MSAlg}(C_1)$. Then $H\#x = \langle F(g), F(f)\rangle$.

(41) For every category C and for every object a of C holds $(\mathrm{Den}(\mathrm{idsym}\,a, \mathrm{MSAlg}(C)))(\emptyset) = \mathrm{id}_a$.

(42) Let C be a category, a, b, c be objects of C, and f, g be morphisms of C. If $f \in \hom(a,b)$ and $g \in \hom(b,c)$, then $(\mathrm{Den}(\mathrm{compsym}(a,b,c), \mathrm{MSAlg}(C)))(\langle g, f\rangle) = g \cdot f$.

(43) Let C be a category, a, b, c, d be objects of C, and f, g, h be morphisms of C. Suppose $f \in \hom(a,b)$ and $g \in \hom(b,c)$ and $h \in \hom(c,d)$. Then $(\mathrm{Den}(\mathrm{compsym}(a,c,d), \mathrm{MSAlg}(C)))(\langle h, (\mathrm{Den}(\mathrm{compsym}(a,b,c), \mathrm{MSAlg}(C)))(\langle g, f\rangle)\rangle) = (\mathrm{Den}(\mathrm{compsym}(a,b,d), \mathrm{MSAlg}(C)))(\langle(\mathrm{Den}(\mathrm{compsym}(b,c,d), \mathrm{MSAlg}(C)))(\langle h, g\rangle), f\rangle)$.

(44) Let C be a category, a, b be objects of C, and f be a morphism of C. If $f \in \hom(a,b)$, then $(\mathrm{Den}(\mathrm{compsym}(a,b,b), \mathrm{MSAlg}(C)))(\langle \mathrm{id}_b, f\rangle) = f$ and $(\mathrm{Den}(\mathrm{compsym}(a,a,b), \mathrm{MSAlg}(C)))(\langle f, \mathrm{id}_a\rangle) = f$.

(45) Let C_1, C_2 be categories and F be a functor from C_1 to C_2. Then there exists a many sorted function H from $\mathrm{MSAlg}(C_1)$ into $\mathrm{MSAlg}(C_2)\restriction_{(\Upsilon_F,\Psi_F)}$ CatSign(the objects of C_1) such that

(i) $H = F\restriction_{\text{the carrier of CatSign(the objects of } C_1)}$the sorts of $\mathrm{MSAlg}(C_1)$, and

(ii) H is a homomorphism of $\mathrm{MSAlg}(C_1)$ into $\mathrm{MSAlg}(C_2)\restriction_{(\Upsilon_F,\Psi_F)}$ CatSign(the objects of C_1).

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Scott Topology[1]

Andrzej Trybulec
Warsaw University
Białystok

Summary. In the article we continue the formalization in Mizar of [15, 98–105]. We work with structures of the form

$$L = \langle C, \leqslant, \tau \rangle,$$

where C is the carrier of the structure, $\leqslant$ - an ordering relation on C and τ a family of subsets of C. When $\langle C, \leqslant \rangle$ is a complete lattice we say that L is Scott, if τ is the Scott topology of $\langle C, \leqslant \rangle$. We define the Scott convergence (lim inf convergence). Following [15] we prove that in the case of a continuous lattice $\langle C, \leqslant \rangle$ the Scott convergence is topological, i.e. enjoys the properties: (CONSTANTS), (SUBNETS), (DIVERGENCE), (ITERATED LIMITS). We formalize the theorem, that if the Scott convergence has the (ITERATED LIMITS) property, the $\langle C, \leqslant \rangle$ is continuous.

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The terminology and notation used in this paper are introduced in the following articles: [29], [35], [37], [25], [12], [14], [36], [10], [11], [9], [3], [8], [33], [23], [27], [38], [28], [26], [41], [17], [30], [2], [24], [1], [22], [34], [4], [5], [6], [16], [40], [13], [18], [19], [20], [7], [39], [32], [21], and [31].

1. Preliminaries

The scheme *Irrel* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, a binary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

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$\{\mathcal{F}(u), u$ ranges over elements of $\mathcal{A} : \mathcal{P}[u]\} = \{\mathcal{F}(i, v), i$ ranges over elements of $\mathcal{B}, v$ ranges over elements of $\mathcal{A} : \mathcal{P}[v]\}$

provided the following condition is met:

- For every element i of $\mathcal{B}$ and for every element u of $\mathcal{A}$ holds $\mathcal{F}(u) = \mathcal{F}(i, u)$.

One can prove the following three propositions:

(1) Let L be a complete non empty lattice and X, Y be subsets of the carrier of L. If Y is coarser than X, then $\bigsqcap_L X \leqslant \bigsqcap_L Y$.

(2) Let L be a complete non empty lattice and X, Y be subsets of the carrier of L. If X is finer than Y then $\bigsqcup_L X \leqslant \bigsqcup_L Y$.

(3) Let T be a relational structure, A be an upper subset of T, and B be a directed subset of T. Then $A \cap B$ is directed.

Let T be a reflexive non empty relational structure. Observe that there exists a subset of T which is non empty, directed, and finite.

Next we state the proposition

(4) For every non empty poset T with l.u.b.'s and for every non empty directed finite subset D of T holds $\sup D \in D$.

Let us observe that there exists a relational structure which is trivial, reflexive, transitive, non empty, antisymmetric, finite, and strict and has l.u.b.'s.

Let us observe that there exists a 1-sorted structure which is finite, non empty, and strict.

Let T be a finite 1-sorted structure. Note that every subset of T is finite.

Let R be a relational structure. Note that $\emptyset_R$ is lower and upper.

Let R be a trivial non empty relational structure. Note that every subset of R is upper.

One can prove the following propositions:

(5) Let T be a non empty relational structure, x be an element of T, and A be an upper subset of T. If $x \notin A$, then A misses $\downarrow x$.

(6) Let T be a non empty relational structure, x be an element of T, and A be a lower subset of T. If $x \in A$, then $\downarrow x \subseteq A$.

2. Scott Topology

Let T be a non empty reflexive relational structure and let S be a subset of T. We say that S is inaccessible by directed joins if and only if:

(Def. 1) For every non empty directed subset D of T such that $\sup D \in S$ holds D meets S.

We introduce S is inaccessible as a synonym of S is inaccessible by directed joins. We say that S is closed under directed sups if and only if:

(Def. 2) For every non empty directed subset D of T such that $D \subseteq S$ holds $\sup D \in S$.

We introduce S is directly closed as a synonym of S is closed under directed sups.
We say that S is property(S) if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let D be a non empty directed subset of T. Suppose $\sup D \in S$. Then there exists an element y of T such that $y \in D$ and for every element x of T such that $x \in D$ and $x \geqslant y$ holds $x \in S$.

We introduce S has the property (S) as a synonym of S is property(S).

Let T be a non empty reflexive relational structure. One can check that $\emptyset_T$ is property(S) and directly closed.

Let T be a non empty reflexive relational structure. Observe that there exists a subset of T which is property(S) and directly closed.

Let T be a non empty reflexive relational structure and let S be a property(S) subset of T. One can verify that $-S$ is directly closed.

Let T be a reflexive non empty FR-structure. We say that T is Scott if and only if:

(Def. 4) For every subset S of T holds S is open iff S is inaccessible and upper.

Let T be a reflexive transitive antisymmetric non empty finite relational structure with l.u.b.'s. Note that every subset of T is inaccessible.

Let T be a reflexive transitive antisymmetric non empty finite FR-structure with l.u.b.'s. Let us observe that T is Scott if and only if:

(Def. 5) For every subset S of T holds S is open iff S is upper.

Let us mention that there exists a non empty strict TopLattice which is trivial, complete, and Scott.

Let T be a non empty reflexive relational structure. Observe that Ω_T is directly closed and inaccessible.

Let T be a non empty reflexive relational structure. Note that there exists a subset of T which is directly closed, lower, inaccessible, and upper.

Let T be a complete non empty TopLattice and let S be an inaccessible subset of T. Note that $-S$ is directly closed.

Let T be a non empty reflexive relational structure and let S be a directly closed subset of T. One can check that $-S$ is inaccessible.

One can prove the following propositions:

(7) Let T be a complete Scott non empty TopLattice and S be a subset of T. Then S is closed if and only if S is directly closed and lower.

(8) For every complete non empty TopLattice T and for every element x of T holds $\downarrow x$ is directly closed.

(9) For every complete Scott non empty TopLattice T and for every element x of T holds $\overline{\{x\}} = \downarrow x$.

(10) Every complete Scott non empty TopLattice is a T_0-space.

(11) For every complete Scott non empty TopLattice T and for every element x of T holds $\downarrow x$ is closed.

(12) For every complete Scott non empty TopLattice T and for every element x of T holds $-\downarrow x$ is open.

(13) Let T be a complete Scott non empty TopLattice, x be an element of T, and A be an upper subset of T. If $x \notin A$, then $-\downarrow x$ is a neighbourhood of A.

(14) Let T be a complete Scott non empty TopLattice and S be an upper subset of T. Then there exists a family F of subsets of T such that $S = \bigcap F$ and for every subset X of T such that $X \in F$ holds X is a neighbourhood of S.

(15) Let T be a Scott non empty TopLattice and S be a subset of T. Then S is open if and only if S is upper and property(S).

Let T be a complete non empty TopLattice. Observe that every subset of T which is lower is also property(S).

One can prove the following proposition

(16) Let T be a non empty transitive reflexive FR-structure. Suppose the topology of $T = \{S, S$ ranges over subsets of T: S has the property (S)$\}$. Then T is topological space-like.

3. Scott Convergence

In the sequel R will be a non empty relational structure, N will be a net in R, and i, j will be elements of the carrier of N.

Let us consider R, N. The functor $\liminf N$ yielding an element of R is defined by:

(Def. 6) $\liminf N = \bigsqcup_R\{\bigsqcap_R\{N(i) : i \geqslant j\} : j$ ranges over elements of the carrier of $N\}$.

Let R be a reflexive non empty relational structure, let N be a net in R, and let p be an element of the carrier of R. We say that p is S-limit of N if and only if:

(Def. 7) $p \leqslant \liminf N$.

Let R be a reflexive non empty relational structure. The Scott convergence of R yields a convergence class of R and is defined by the condition (Def. 8).

(Def. 8) Let N be a strict net in R. Suppose $N \in \mathrm{NetUniv}(R)$. Let p be an element of the carrier of R. Then $\langle N, p\rangle \in$ the Scott convergence of R if and only if p is S-limit of N.

The following two propositions are true:

(17) Let R be a non empty complete lattice, N be a net in R, and p, q be elements of the carrier of R. If p is S-limit of N and N is eventually in $\downarrow q$, then $p \leqslant q$.

(18) Let R be a non empty complete lattice, N be a net in R, and p, q be elements of the carrier of R. If N is eventually in $\uparrow q$, then $\liminf N \geqslant q$.

Let R be a reflexive non empty relational structure and let N be a non empty net structure over R. Let us observe that N is monotone if and only if:

(Def. 9) For all elements i, j of the carrier of N such that $i \leqslant j$ holds $N(i) \leqslant N(j)$.

Let R be a non empty relational structure, let S be a non empty set, and let f be a function from S into the carrier of R. The functor $\mathrm{NetStr}(S, f)$ yielding a strict non empty net structure over R is defined by the conditions (Def. 10).

(Def. 10)(i) The carrier of $\mathrm{NetStr}(S, f) = S$,

(ii) the mapping of $\mathrm{NetStr}(S, f) = f$, and

(iii) for all elements i, j of $\mathrm{NetStr}(S, f)$ holds $i \leqslant j$ iff $(\mathrm{NetStr}(S, f))(i) \leqslant (\mathrm{NetStr}(S, f))(j)$.

The following two propositions are true:

(19) Let L be a non empty 1-sorted structure and N be a non empty net structure over L. Then $\mathrm{rng}\,(\text{the mapping of } N) = \{N(i) : i \text{ ranges over elements of the carrier of } N\}$.

(20) Let R be a non empty relational structure, S be a non empty set, and f be a function from S into the carrier of R. If $\mathrm{rng}\, f$ is directed, then $\mathrm{NetStr}(S, f)$ is directed.

Let R be a non empty relational structure, let S be a non empty set, and let f be a function from S into the carrier of R. Note that $\mathrm{NetStr}(S, f)$ is monotone.

Let R be a transitive non empty relational structure, let S be a non empty set, and let f be a function from S into the carrier of R. Note that $\mathrm{NetStr}(S, f)$ is transitive.

Let R be a reflexive non empty relational structure, let S be a non empty set, and let f be a function from S into the carrier of R. Observe that $\mathrm{NetStr}(S, f)$ is reflexive.

We now state the proposition

(21) Let R be a non empty transitive relational structure, S be a non empty set, and f be a function from S into the carrier of R. If $S \subseteq$ the carrier of R and $\mathrm{NetStr}(S, f)$ is directed, then $\mathrm{NetStr}(S, f) \in \mathrm{NetUniv}(R)$.

Let R be a non empty lattice. One can check that there exists a net in R which is monotone, reflexive, and strict.

The following propositions are true:

(22) For every non empty complete lattice R and for every monotone reflexive net N in R holds $\liminf N = \sup N$.

(23) For every complete non empty lattice R and for every constant net N in R holds the value of $N = \liminf N$.

(24) For every complete non empty lattice R and for every constant net N in R holds the value of N is S-limit of N.

Let S be a non empty 1-sorted structure and let e be an element of the carrier of S. The functor $\mathrm{NetStr}(e)$ yielding a strict net structure over S is defined as follows:

(Def. 11) The carrier of $\mathrm{NetStr}(e) = \{e\}$ and the internal relation of $\mathrm{NetStr}(e) = \{\langle e, e\rangle\}$ and the mapping of $\mathrm{NetStr}(e) = \mathrm{id}_{\{e\}}$.

Let S be a non empty 1-sorted structure and let e be an element of the carrier of S. Observe that $\mathrm{NetStr}(e)$ is non empty.

One can prove the following propositions:

(25) Let S be a non empty 1-sorted structure, e be an element of the carrier of S, and x be an element of $\mathrm{NetStr}(e)$. Then $x = e$.

(26) Let S be a non empty 1-sorted structure, e be an element of the carrier of S, and x be an element of $\mathrm{NetStr}(e)$. Then $(\mathrm{NetStr}(e))(x) = e$.

Let S be a non empty 1-sorted structure and let e be an element of the carrier of S. Observe that $\mathrm{NetStr}(e)$ is reflexive transitive directed and antisymmetric.

We now state several propositions:

(27) Let S be a non empty 1-sorted structure, e be an element of the carrier of S, and X be a set. Then $\mathrm{NetStr}(e)$ is eventually in X if and only if $e \in X$.

(28) Let S be a reflexive antisymmetric non empty relational structure and e be an element of the carrier of S. Then $e = \liminf \mathrm{NetStr}(e)$.

(29) For every non empty reflexive relational structure S and for every element e of the carrier of S holds $\mathrm{NetStr}(e) \in \mathrm{NetUniv}(S)$.

(30) Let R be a non empty complete lattice, Z be a net in R, and D be a subset of R. Suppose $D = \{\bigsqcap_R\{Z(k), k$ ranges over elements of the carrier of Z: $k \geqslant j\} : j$ ranges over elements of the carrier of $Z\}$. Then D is non empty and directed.

(31) Let L be a non empty complete lattice and S be a subset of L. Then $S \in$ the topology of ConvergenceSpace(the Scott convergence of L) if and only if S is inaccessible and upper.

(32) Let T be a non empty complete Scott TopLattice. Then the topological structure of $T =$ ConvergenceSpace(the Scott convergence of T).

(33) Let T be a non empty complete TopLattice. Suppose the topological structure of $T =$ ConvergenceSpace(the Scott convergence of T). Let S be a subset of T. Then S is open if and only if S is inaccessible and upper.

(34) Let T be a non empty complete TopLattice. Suppose the topological structure of $T =$ ConvergenceSpace(the Scott convergence of T). Then T is Scott.

Let R be a complete non empty lattice. Note that the Scott convergence of R has (CONSTANTS) property.

Let R be a complete non empty lattice. Observe that the Scott convergence of R has (SUBNETS) property.

The following proposition is true

(35) Let S be a non empty 1-sorted structure, N be a net in S, X be a set, and M be a subnet of N. If $M = N^{-1}(X)$, then for every element i of the carrier of M holds $M(i) \in X$.

Let L be a non empty complete lattice. The functor $\mathrm{sigma}\, L$ yielding a family of subsets of L is defined as follows:

(Def. 12) $\mathrm{sigma}\, L =$ the topology of ConvergenceSpace(the Scott convergence of L).

One can prove the following propositions:

(36) For every continuous complete Scott TopLattice L and for every element x of L holds $\uparrow x$ is open.

(37) For every non empty complete TopLattice T such that the topology of $T = \operatorname{sigma} T$ holds T is Scott.

Let R be a continuous non empty complete lattice. Observe that the Scott convergence of R is topological.

We now state a number of propositions:

(38) Let T be a continuous non empty complete Scott TopLattice, x be an element of the carrier of T, and N be a net in T. If $N \in \operatorname{NetUniv}(T)$, then x is S-limit of N iff $x \in \operatorname{Lim} N$.

(39) Let L be a complete non empty poset. Suppose the Scott convergence of L has (ITERATED LIMITS) property. Then L is continuous.

(40) Let T be a complete Scott non empty TopLattice. Then T is continuous if and only if $\operatorname{Convergence}(T) =$ the Scott convergence of T.

(41)[2] For every complete Scott non empty TopLattice T and for every upper subset S of T such that S is open holds S is open.

(42) Let L be a non empty relational structure, S be an upper subset of L, and x be an element of L. If $x \in S$, then $\uparrow x \subseteq S$.

(43) Let L be a non empty complete continuous Scott TopLattice, p be an element of L, and S be a subset of L. If S is open and $p \in S$, then there exists an element q of L such that $q \ll p$ and $q \in S$.

(44) Let L be a non empty complete continuous Scott TopLattice and p be an element of L. Then $\{\Uparrow q, q$ ranges over elements of L: $q \ll p\}$ is a basis of p.

(45) For every complete continuous Scott non empty TopLattice T holds $\{\Uparrow x :$ x ranges over elements of $T\}$ is a basis of T.

(46)[3] Let T be a complete continuous Scott non empty TopLattice and S be an upper subset of T. Then S is open if and only if S is open.

(47) For every complete continuous Scott non empty TopLattice T and for every element p of T holds $\operatorname{Int}\uparrow p = \Uparrow p$.

(48) Let T be a complete continuous Scott non empty TopLattice and S be a subset of T. Then $\operatorname{Int} S = \bigcup\{\Uparrow x, x$ ranges over elements of T: $\Uparrow x \subseteq S\}$.

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On the Baire Category Theorem[1]

Artur Korniłowicz
Warsaw University
Białystok

Summary. In this paper Exercise 3.43 from Chapter 1 of [14] is solved.

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The terminology and notation used in this paper have been introduced in the following articles: [23], [27], [2], [28], [10], [11], [8], [13], [25], [9], [1], [4], [21], [26], [29], [12], [17], [22], [3], [5], [16], [6], [30], [18], [19], [7], [15], [20], and [24].

1. Preliminaries

Let T be a topological structure and let A be a subset of the carrier of T. Then $\operatorname{Int} A$ is a subset of T.

Let T be a topological structure and let P be a subset of the carrier of T. Let us observe that P is closed if and only if:

(Def. 1) $-P$ is open.

Let T be a non empty topological space and let F be a family of subsets of T. We say that F is dense if and only if:

(Def. 2) For every subset X of T such that $X \in F$ holds X is dense.

The following proposition is true

(1) Let L be a non empty 1-sorted structure, A be a subset of L, and x be an element of L. Then $x \in -A$ if and only if $x \notin A$.

Let us observe that there exists a 1-sorted structure which is empty.

Let S be an empty 1-sorted structure. Note that the carrier of S is empty.

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Let S be an empty 1-sorted structure. Note that every subset of S is empty.

One can check that every set which is finite is also countable.

Let us note that there exists a set which is empty.

Let S be a 1-sorted structure. One can verify that there exists a subset of S which is empty.

One can verify that there exists a set which is non empty and finite.

Let L be a non empty relational structure. Observe that there exists a subset of L which is non empty and finite.

Let us note that $\mathbb{N}$ is infinite.

Let us note that there exists a set which is infinite and countable.

Let S be a 1-sorted structure. One can verify that there exists a family of subsets of S which is empty.

One can prove the following propositions:

(2) For all sets X, Y such that $\overline{\overline{X}} \leqslant \overline{\overline{Y}}$ and Y is countable holds X is countable.

(3) For every infinite countable set A holds $\mathbb{N} \approx A$.

(4) For every non empty countable set A there exists a function f from $\mathbb{N}$ into A such that $\operatorname{rng} f = A$.

(5) For every 1-sorted structure S and for all subsets X, Y of S holds $-(X \cup Y) = (-X) \cap -Y$.

(6) For every 1-sorted structure S and for all subsets X, Y of S holds $-X \cap Y = -X \cup -Y$.

(7) Let L be a non empty transitive relational structure and A, B be subsets of L. If A is finer than B, then $\downarrow A \subseteq \downarrow B$.

(8) Let L be a non empty transitive relational structure and A, B be subsets of L. If A is coarser than B, then $\uparrow A \subseteq \uparrow B$.

(9) Let L be a non empty poset and D be a non empty finite filtered subset of L. If $\inf D$ exists in L, then $\inf D \in D$.

(10) Let L be a lower-bounded antisymmetric non empty relational structure and X be a non empty lower subset of L. Then $\perp_L \in X$.

(11) Let L be a lower-bounded antisymmetric non empty relational structure and X be a non empty subset of L. Then $\perp_L \in \downarrow X$.

(12) Let L be an upper-bounded antisymmetric non empty relational structure and X be a non empty upper subset of L. Then $\top_L \in X$.

(13) Let L be an upper-bounded antisymmetric non empty relational structure and X be a non empty subset of L. Then $\top_L \in \uparrow X$.

(14) Let L be a lower-bounded antisymmetric relational structure with g.l.b.'s and X be a subset of L. Then $X \sqcap \{\perp_L\} \subseteq \{\perp_L\}$.

(15) Let L be a lower-bounded antisymmetric relational structure with g.l.b.'s and X be a non empty subset of L. Then $X \sqcap \{\perp_L\} = \{\perp_L\}$.

(16) Let L be an upper-bounded antisymmetric relational structure with l.u.b.'s and X be a subset of L. Then $X \sqcup \{\top_L\} \subseteq \{\top_L\}$.

(17) Let L be an upper-bounded antisymmetric relational structure with l.u.b.'s and X be a non empty subset of L. Then $X \sqcup \{\top_L\} = \{\top_L\}$.

(18) For every upper-bounded semilattice L and for every subset X of L holds $\{\top_L\} \sqcap X = X$.

(19) For every lower-bounded poset L with l.u.b.'s and for every subset X of L holds $\{\perp_L\} \sqcup X = X$.

(20) Let L be a non empty reflexive relational structure and A, B be subsets of L. If $A \subseteq B$, then A is finer than B and coarser than B.

(21) Let L be an antisymmetric transitive relational structure with g.l.b.'s, V be a subset of L, and x, y be elements of L. If $x \leqslant y$, then $\{y\} \sqcap V$ is coarser than $\{x\} \sqcap V$.

(22) Let L be an antisymmetric transitive relational structure with l.u.b.'s, V be a subset of L, and x, y be elements of L. If $x \leqslant y$, then $\{x\} \sqcup V$ is finer than $\{y\} \sqcup V$.

(23) Let L be a non empty relational structure and V, S, T be subsets of L. If S is coarser than T and V is upper and $T \subseteq V$, then $S \subseteq V$.

(24) Let L be a non empty relational structure and V, S, T be subsets of L. If S is finer than T and V is lower and $T \subseteq V$, then $S \subseteq V$.

(25) For every semilattice L and for every upper filtered subset F of L holds $F \sqcap F = F$.

(26) For every sup-semilattice L and for every lower directed subset I of L holds $I \sqcup I = I$.

(27) For every upper-bounded semilattice L and for every subset V of L holds $\{x, x$ ranges over elements of L: $V \sqcap \{x\} \subseteq V\}$ is non empty.

(28) Let L be an antisymmetric transitive relational structure with g.l.b.'s and V be a subset of L. Then $\{x, x$ ranges over elements of L: $V \sqcap \{x\} \subseteq V\}$ is a filtered subset of L.

(29) Let L be an antisymmetric transitive relational structure with g.l.b.'s and V be an upper subset of L. Then $\{x, x$ ranges over elements of L: $V \sqcap \{x\} \subseteq V\}$ is an upper subset of L.

(30) For every poset L with g.l.b.'s and for every subset X of L such that X is open and lower holds X is filtered.

Let L be a poset with g.l.b.'s. Observe that every subset of L which is open and lower is also filtered.

Let L be a continuous antisymmetric non empty reflexive relational structure. One can verify that every subset of L which is lower is also open.

Let L be a continuous semilattice and let x be an element of L. Note that $-{\downarrow}x$ is open.

We now state two propositions:

(31) Let L be a semilattice and C be a non empty subset of L. Suppose that for all elements x, y of L such that $x \in C$ and $y \in C$ holds $x \leqslant y$ or $y \leqslant x$. Let Y be a non empty finite subset of C. Then $\bigsqcap_L Y \in Y$.

(32) Let L be a sup-semilattice and C be a non empty subset of L. Suppose that for all elements x, y of L such that $x \in C$ and $y \in C$ holds $x \leqslant y$ or $y \leqslant x$. Let Y be a non empty finite subset of C. Then $\bigsqcup_L Y \in Y$.

Let L be a semilattice and let F be a filter of L. A subset of L is called a generator set of F if:

(Def. 3) $F = \uparrow\mathrm{fininfs}(\mathrm{it})$.

Let L be a semilattice and let F be a filter of L. One can verify that there exists a generator set of F which is non empty.

The following propositions are true:

(33) Let L be a semilattice, A be a subset of L, and B be a non empty subset of L. If A is coarser than B, then $\mathrm{fininfs}(A)$ is coarser than $\mathrm{fininfs}(B)$.

(34) Let L be a semilattice, F be a filter of L, G be a generator set of F, and A be a non empty subset of L. Suppose G is coarser than A and A is coarser than F. Then A is a generator set of F.

(35) Let L be a semilattice, A be a subset of L, and f, g be functions from $\mathbb{N}$ into the carrier of L. Suppose $\mathrm{rng}\, f = A$ and for every element n of $\mathbb{N}$ holds $g(n) = \bigsqcap_L \{f(m), m$ ranges over natural numbers: $m \leqslant n\}$. Then A is coarser than $\mathrm{rng}\, g$.

(36) Let L be a semilattice, F be a filter of L, G be a generator set of F, and f, g be functions from $\mathbb{N}$ into the carrier of L. Suppose $\mathrm{rng}\, f = G$ and for every element n of $\mathbb{N}$ holds $g(n) = \bigsqcap_L \{f(m), m$ ranges over natural numbers: $m \leqslant n\}$. Then $\mathrm{rng}\, g$ is a generator set of F.

2. On the Baire Category Theorem

The following propositions are true:

(37) Let L be a lower-bounded continuous lattice, V be an open upper subset of L, F be a filter of L, and v be an element of L. Suppose $V \sqcap F \subseteq V$ and $v \in V$ and there exists a non empty generator set of F which is countable. Then there exists an open filter O of L such that $O \subseteq V$ and $v \in O$ and $F \subseteq O$.

(38) Let L be a lower-bounded continuous lattice, V be an open upper subset of L, N be a non empty countable subset of L, and v be an element of L. Suppose $V \sqcap N \subseteq V$ and $v \in V$. Then there exists an open filter O of L such that $\{v\} \sqcap N \subseteq O$ and $O \subseteq V$ and $v \in O$.

(39) Let L be a lower-bounded continuous lattice, V be an open upper subset of L, N be a non empty countable subset of L, and x, y be elements of L. Suppose $V \sqcap N \subseteq V$ and $y \in V$ and $x \notin V$. Then there exists an irreducible element p of L such that $x \leqslant p$ and $p \notin \uparrow(\{y\} \sqcap N)$.

(40) Let L be a lower-bounded continuous lattice, x be an element of L, and N be a non empty countable subset of L. Suppose that for all elements n, y of L such that $y \nleqslant x$ and $n \in N$ holds $y \sqcap n \nleqslant x$. Let y be an element

of L. Suppose $y \nleqslant x$. Then there exists an irreducible element p of L such that $x \leqslant p$ and $p \notin \uparrow(\{y\} \sqcap N)$.

Let L be a non empty relational structure and let u be an element of L. We say that u is dense if and only if:

(Def. 4) For every element v of L such that $v \neq \bot_L$ holds $u \sqcap v \neq \bot_L$.

Let L be an upper-bounded semilattice. Note that $\top_L$ is dense.

Let L be an upper-bounded semilattice. Note that there exists an element of L which is dense.

The following proposition is true

(41) For every non trivial bounded semilattice L and for every element x of L such that x is dense holds $x \neq \bot_L$.

Let L be a non empty relational structure and let D be a subset of L. We say that D is dense if and only if:

(Def. 5) For every element d of L such that $d \in D$ holds d is dense.

We now state the proposition

(42) For every upper-bounded semilattice L holds $\{\top_L\}$ is dense.

Let L be an upper-bounded semilattice. Note that there exists a subset of L which is non empty, finite, countable, and dense.

Next we state several propositions:

(43) Let L be a lower-bounded continuous lattice, D be a non empty countable dense subset of L, and u be an element of L. Suppose $u \neq \bot_L$. Then there exists an irreducible element p of L such that $p \neq \top_L$ and $p \notin \uparrow(\{u\} \sqcap D)$.

(44) Let T be a non empty topological space, A be an element of $\langle$the topology of T, $\subseteq\rangle$, and B be a subset of T. If $A = B$ and $-B$ is irreducible, then A is irreducible.

(45) Let T be a non empty topological space, A be an element of $\langle$the topology of T, $\subseteq\rangle$, and B be a subset of T. Suppose $A = B$ and $A \neq \top_{\langle \text{the topology of } T,\ \subseteq\rangle}$. Then A is irreducible if and only if $-B$ is irreducible.

(46) Let T be a non empty topological space, A be an element of $\langle$the topology of T, $\subseteq\rangle$, and B be a subset of T. If $A = B$, then A is dense iff B is everywhere dense.

(47) Let T be a non empty topological space. Suppose T is locally-compact. Let D be a countable family of subsets of T. Suppose D is non empty, dense, and open. Let O be a non empty subset of T. Suppose O is open. Then there exists an irreducible subset A of T such that for every subset V of T if $V \in D$, then $A \cap O \cap V \neq \emptyset$.

Let T be a non empty topological space. Let us observe that T is Baire if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let F be a family of subsets of T. Suppose F is countable and for every subset S of T such that $S \in F$ holds S is open and dense. Then $\mathrm{Intersect}(F)$ is dense.

Next we state the proposition

(48) For every non empty topological space T such that T is sober and locally-compact holds T is Baire.

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