FORMALIZED MATHEMATICS
Volume 6, Number 1, 1997
Warsaw University - Białystok

Properties of Relational Structures, Posets, Lattices and Maps [1]

Mariusz Żynel
Warsaw University
Białystok

Czesław Byliński
Warsaw University
Białystok

Summary. In the paper we present some auxiliary facts concerning posets and maps between them. Our main purpose, however is to give an account on complete lattices and lattices of ideals. A sufficient condition that a lattice might be complete, the fixed-point theorem and two remarks upon images of complete lattices in monotone maps, introduced in [10, pp. 8–9], can be found in Section 7. Section 8 deals with lattices of ideals. We examine the meet and join of two ideals. In order to show that the lattice of ideals is complete, the infinite intersection of ideals is investigated.

MML Identifier: `YELLOW_2`.

The terminology and notation used in this paper have been introduced in the following articles: [18], [20], [21], [7], [8], [2], [17], [15], [19], [3], [6], [13], [16], [9], [14], [5], [11], [1], [12], and [4].

1. Basic Facts

In this paper x will be arbitrary and X, Y will denote sets.

The scheme *RelStrSubset* deals with a non empty relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

$\{x : x$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[x]\}$ is a subset of $\mathcal{A}$

for all values of the parameters.

Let S be a non empty 1-sorted structure and let X be a non empty subset of the carrier of S. We see that the element of X is an element of S.

One can prove the following four propositions:

[1]This work was partially supported by Office of Naval Research Grant N00014-95-1-1336.

ISSN 1426–2630

(1) Let L be a non empty relational structure, and let x be an element of L, and let X be a subset of L. Then $X \subseteq \downarrow x$ if and only if $X \leq x$.

(2) Let L be a non empty relational structure, and let x be an element of L, and let X be a subset of L. Then $X \subseteq \uparrow x$ if and only if $x \leq X$.

(3) Let L be an antisymmetric transitive relational structure with l.u.b.'s and let X, Y be sets. Suppose sup X exists in L and sup Y exists in L. Then sup $X \cup Y$ exists in L and $\bigsqcup_L (X \cup Y) = \bigsqcup_L X \sqcup \bigsqcup_L Y$.

(4) Let L be an antisymmetric transitive relational structure with g.l.b.'s and let X, Y be sets. Suppose inf X exists in L and inf Y exists in L. Then inf $X \cup Y$ exists in L and $\bigsqcap_L (X \cup Y) = \bigsqcap_L X \sqcap \bigsqcap_L Y$.

2. Relational Substructures

The following propositions are true:

(5) For every binary relation R and for all sets X, Y such that $X \subseteq Y$ holds $R\,|^2\, X \subseteq R\,|^2\, Y$.

(6) Let L be a relational structure and let S, T be full relational substructures of L. Suppose the carrier of $S \subseteq$ the carrier of T. Then the internal relation of $S \subseteq$ the internal relation of T.

(7) Let L be a non empty relational structure and let S be a non empty relational substructure of L. Then

(i) if X is a directed subset of S, then X is a directed subset of L, and

(ii) if X is a filtered subset of S, then X is a filtered subset of L.

(8) Let L be a non empty relational structure and let S, T be non empty full relational substructures of L. Suppose the carrier of $S \subseteq$ the carrier of T. Let X be a subset of S. Then

(i) X is a subset of T, and

(ii) for every subset Y of T such that $X = Y$ holds if X is filtered, then Y is filtered and if X is directed, then Y is directed.

3. Maps

Now we present three schemes. The scheme *LambdaMD* deals with non empty relational structures $\mathcal{A}$, $\mathcal{B}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

> There exists a map f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

The scheme *KappaMD* concerns non empty relational structures $\mathcal{A}$, $\mathcal{B}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

There exists a map f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

provided the parameters satisfy the following condition:

- For every element x of $\mathcal{A}$ holds $\mathcal{F}(x)$ is an element of $\mathcal{B}$.

The scheme *NonUniqExMD* deals with non empty relational structures $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a map f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the following requirement is met:

- For every element x of $\mathcal{A}$ there exists an element y of $\mathcal{B}$ such that $\mathcal{P}[x, y]$.

Let S, T be 1-sorted structures and let f be a map from S into T. Then $\operatorname{rng} f$ is a subset of T.

One can prove the following proposition

(9) Let S, T be non empty 1-sorted structures and let f, g be maps from S into T. If for every element s of S holds $f(s) = g(s)$, then $f = g$.

Let J be a set, let L be a relational structure, and let f, g be functions from J into the carrier of L. The predicate $f \leq g$ is defined by:

(Def. 1) For arbitrary j such that $j \in J$ there exist elements a, b of L such that $a = f(j)$ and $b = g(j)$ and $a \leq b$.

We introduce $g \geq f$ as a synonym of $f \leq g$.

Next we state the proposition

(10) Let L, M be non empty relational structures and let f, g be maps from L into M. Then $f \leq g$ if and only if for every element x of L holds $f(x) \leq g(x)$.

4. The Image of a Map

Let L, M be non empty relational structures and let f be a map from L into M. The functor $\operatorname{Im} f$ yields a strict full relational substructure of M and is defined as follows:

(Def. 2) $\operatorname{Im} f = \operatorname{sub}(\operatorname{rng} f)$.

The following two propositions are true:

(11) For all non empty relational structures L, M and for every map f from L into M holds $\operatorname{rng} f =$ the carrier of $\operatorname{Im} f$.

(12) Let L, M be non empty relational structures, and let f be a map from L into M, and let y be an element of $\operatorname{Im} f$. Then there exists an element x of L such that $f(x) = y$.

Let L be a non empty relational structure and let X be a non empty subset of L. One can verify that $\operatorname{sub}(X)$ is non empty.

Let L, M be non empty relational structures and let f be a map from L into M. Observe that $\operatorname{Im} f$ is non empty.

5. Monotone Maps

One can prove the following propositions:

(13) For every non empty relational structure L holds id_L is monotone.

(14) Let L, M, N be non empty relational structures, and let f be a map from L into M, and let g be a map from M into N. If f is monotone and g is monotone, then $g \cdot f$ is monotone.

(15) Let L, M be non empty relational structures, and let f be a map from L into M, and let X be a subset of L, and let x be an element of L. If f is monotone and $x \leq X$, then $f(x) \leq f^\circ X$.

(16) Let L, M be non empty relational structures, and let f be a map from L into M, and let X be a subset of L, and let x be an element of L. If f is monotone and $X \leq x$, then $f^\circ X \leq f(x)$.

(17) Let S, T be non empty relational structures, and let f be a map from S into T, and let X be a directed subset of S. If f is monotone, then $f^\circ X$ is directed.

(18) Let L be a poset with l.u.b.'s and let f be a map from L into L. If f is directed-sups-preserving, then f is monotone.

(19) Let L be a poset with g.l.b.'s and let f be a map from L into L. If f is filtered-infs-preserving, then f is monotone.

6. Idempotent Maps

One can prove the following propositions:

(20) Let S be a non empty 1-sorted structure and let f be a map from S into S. If f is idempotent, then for every element x of S holds $f(f(x)) = f(x)$.

(21) Let S be a non empty 1-sorted structure and let f be a map from S into S. If f is idempotent, then $\mathrm{rng}\, f = \{x : x$ ranges over elements of S, $x = f(x)\}$.

(22) Let S be a non empty 1-sorted structure and let f be a map from S into S. If f is idempotent, then if $X \subseteq \mathrm{rng}\, f$, then $f^\circ X = X$.

(23) For every non empty relational structure L holds id_L is idempotent.

7. Complete Lattices

In the sequel L denotes a complete lattice and a denotes an element of L.

The following propositions are true:

(24) If $a \in X$, then $a \leq \bigsqcup_L X$ and $\bigsqcap_L X \leq a$.

(25) Let L be a non empty relational structure. Then for every X holds sup X exists in L if and only if for every Y holds inf Y exists in L.

(26) For every non empty relational structure L such that for every X holds sup X exists in L holds L is complete.

(27) For every non empty relational structure L such that for every X holds inf X exists in L holds L is complete.

(28) Let L be a non empty relational structure. Suppose that for every subset A of L holds inf A exists in L. Given X. Then inf X exists in L and $\bigsqcap_L X = \bigsqcap_L (X \cap \text{(the carrier of } L))$.

(29) Let L be a non empty relational structure. Suppose that for every subset A of L holds sup A exists in L. Given X. Then sup X exists in L and $\bigsqcup_L X = \bigsqcup_L (X \cap \text{(the carrier of } L))$.

(30) Let L be a non empty relational structure. If for every subset A of L holds inf A exists in L, then L is complete.

One can check that every non empty poset which is up-complete, inf-complete, and upper-bounded is also complete.

Next we state several propositions:

(31) Let f be a map from L into L. Suppose f is monotone. Let M be a subset of L. If $M = \{x : x \text{ ranges over elements of } L,\ x = f(x)\}$, then $\operatorname{sub}(M)$ is a complete lattice.

(32) Every infs-inheriting non empty full relational substructure of L is a complete lattice.

(33) Every sups-inheriting non empty full relational substructure of L is a complete lattice.

(34) Let M be a non empty relational structure and let f be a map from L into M. If f is sups-preserving, then $\operatorname{Im} f$ is sups-inheriting.

(35) Let M be a non empty relational structure and let f be a map from L into M. If f is infs-preserving, then $\operatorname{Im} f$ is infs-inheriting.

(36) Let L, M be complete lattices and let f be a map from L into M. Suppose f is sups-preserving or infs-preserving. Then $\operatorname{Im} f$ is a complete lattice.

(37) Let f be a map from L into L. Suppose f is idempotent and directed-sups-preserving. Then $\operatorname{Im} f$ is directed-sups-inheriting and $\operatorname{Im} f$ is a complete lattice.

8. Lattices of Ideals

Next we state several propositions:

(38) Let L be a relational structure and let F be a subset of $2^{\text{the carrier of } L}$. Suppose that for every subset X of L such that $X \in F$ holds X is lower. Then $\bigcap F$ is a lower subset of L.

(39) Let L be a relational structure and let F be a subset of $2^{\text{the carrier of } L}$. Suppose that for every subset X of L such that $X \in F$ holds X is upper. Then $\bigcap F$ is an upper subset of L.

(40) Let L be an antisymmetric relational structure with l.u.b.'s and let F be a subset of $2^{\text{the carrier of } L}$. Suppose that for every subset X of L such that $X \in F$ holds X is lower and directed. Then $\bigcap F$ is a directed subset of L.

(41) Let L be an antisymmetric relational structure with g.l.b.'s and let F be a subset of $2^{\text{the carrier of } L}$. Suppose that for every subset X of L such that $X \in F$ holds X is upper and filtered. Then $\bigcap F$ is a filtered subset of L.

(42) For every poset L with g.l.b.'s and for all ideals I, J of L holds $I \cap J$ is an ideal of L.

Let L be a non empty reflexive transitive relational structure. One can check that $\mathrm{Ids}(L)$ is non empty.

We now state three propositions:

(43) Let L be a non empty reflexive transitive relational structure. Then x is an element of $\langle \mathrm{Ids}(L), \subseteq \rangle$ if and only if x is an ideal of L.

(44) Let L be a non empty reflexive transitive relational structure and let I be an element of $\langle \mathrm{Ids}(L), \subseteq \rangle$. If $x \in I$, then x is an element of L.

(45) For every poset L with g.l.b.'s and for all elements x, y of $\langle \mathrm{Ids}(L), \subseteq \rangle$ holds $x \sqcap y = x \cap y$.

Let L be a poset with g.l.b.'s. One can verify that $\langle \mathrm{Ids}(L), \subseteq \rangle$ has g.l.b.'s.

We now state the proposition

(46) Let L be a poset with l.u.b.'s and let x, y be elements of $\langle \mathrm{Ids}(L), \subseteq \rangle$. Then there exists a subset Z of L such that
  (i) $Z = \{z : z$ ranges over elements of L, $z \in x \ \vee \ z \in y \ \vee \ \bigvee_{a,b\,:\,\text{element of } L} \ a \in x \ \wedge \ b \in y \ \wedge \ z = a \sqcup b\}$,
  (ii) $\sup \{x, y\}$ exists in $\langle \mathrm{Ids}(L), \subseteq \rangle$, and
  (iii) $x \sqcup y = {\downarrow} Z$.

Let L be a poset with l.u.b.'s. One can check that $\langle \mathrm{Ids}(L), \subseteq \rangle$ has l.u.b.'s.

One can prove the following four propositions:

(47) For every lower-bounded sup-semilattice L and for every non empty subset X of $\mathrm{Ids}(L)$ holds $\bigcap X$ is an ideal of L.

(48) Let L be a lower-bounded sup-semilattice and let A be a non empty subset of $\langle \mathrm{Ids}(L), \subseteq \rangle$. Then $\inf A$ exists in $\langle \mathrm{Ids}(L), \subseteq \rangle$ and $\inf A = \bigcap A$.

(49) For every poset L with l.u.b.'s holds $\inf \emptyset$ exists in $\langle \mathrm{Ids}(L), \subseteq \rangle$ and $\bigsqcap_{(\langle \mathrm{Ids}(L), \subseteq \rangle)} \emptyset = \Omega_L$.

(50) For every lower-bounded sup-semilattice L holds $\langle \mathrm{Ids}(L), \subseteq \rangle$ is complete.

Let L be a lower-bounded sup-semilattice. Note that $\langle \mathrm{Ids}(L), \subseteq \rangle$ is complete.

9. Special Maps

Let L be a non empty poset. The functor $\mathrm{SupMap}(L)$ yielding a map from $\langle \mathrm{Ids}(L), \subseteq \rangle$ into L is defined as follows:

(Def. 3) For every ideal I of L holds $(\mathrm{SupMap}(L))(I) = \sup I$.

We now state three propositions:

(51) For every non empty poset L holds $\mathrm{dom}\,\mathrm{SupMap}(L) = \mathrm{Ids}(L)$ and $\mathrm{rng}\,\mathrm{SupMap}(L)$ is a subset of L.

(52) For every non empty poset L holds $x \in \mathrm{dom}\,\mathrm{SupMap}(L)$ iff x is an ideal of L.

(53) For every up-complete non empty poset L holds $\mathrm{SupMap}(L)$ is monotone.

Let L be an up-complete non empty poset. Observe that $\mathrm{SupMap}(L)$ is monotone.

Let L be a non empty poset. The functor $\mathrm{IdsMap}(L)$ yielding a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ is defined by:

(Def. 4) For every element x of L holds $(\mathrm{IdsMap}(L))(x) = {\downarrow} x$.

The following proposition is true

(54) For every non empty poset L holds $\mathrm{IdsMap}(L)$ is monotone.

Let L be a non empty poset. Observe that $\mathrm{IdsMap}(L)$ is monotone.

10. The Family of Elements in a Lattice

Let L be a non empty relational structure and let F be a binary relation. The functor $\bigsqcup_L F$ yielding an element of L is defined as follows:

(Def. 5) $\bigsqcup_L F = \bigsqcup_L \mathrm{rng}\, F$.

The functor $\bigsqcap_L F$ yields an element of L and is defined by:

(Def. 6) $\bigsqcap_L F = \bigsqcap_L \mathrm{rng}\, F$.

Let J be a set, let L be a non empty relational structure, and let F be a function from J into the carrier of L. We introduce $\mathrm{Sup}(F)$ as a synonym of $\bigsqcup_L F$. We introduce $\mathrm{Inf}(F)$ as a synonym of $\bigsqcap_L F$.

Let J be a non empty set, let S be a non empty 1-sorted structure, let F be a function from J into the carrier of S, and let j be an element of J. Then $F(j)$ is an element of S.

Let J be a set, let S be a non empty 1-sorted structure, and let F be a function from J into the carrier of S. Then $\mathrm{rng}\, F$ is a subset of S.

In the sequel J is a non empty set and j is an element of J.

We now state three propositions:

(55) For every function F from J into the carrier of L holds $F(j) \leq \mathrm{Sup}(F)$ and $\mathrm{Inf}(F) \leq F(j)$.

(56) For every function F from J into the carrier of L such that for every j holds $F(j) \leq a$ holds $\mathrm{Sup}(F) \leq a$.

(57) For every function F from J into the carrier of L such that for every j holds $a \leq F(j)$ holds $a \leq \mathrm{Inf}(F)$.

References

[1] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.

[2] Grzegorz Bancerek. Cartesian product of functions. *Formalized Mathematics*, 2(**4**):547–552, 1991.

[3] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.

[4] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.

[5] Grzegorz Bancerek. Quantales. *Formalized Mathematics*, 5(**1**):85–91, 1996.

[6] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(**1**):123–129, 1990.

[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[9] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[10] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[11] Adam Grabowski. On the category of posets. *Formalized Mathematics*, 5(**4**):501–505, 1996.

[12] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Formalized Mathematics*, 6(**1**):117–121, 1997.

[13] Krzysztof Hryniewiecki. Relations of tolerance. *Formalized Mathematics*, 2(**1**):105–109, 1991.

[14] Zbigniew Karno. Continuity of mappings over the union of subspaces. *Formalized Mathematics*, 3(**1**):1–16, 1992.

[15] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[16] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[17] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[19] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.

[20] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[21] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received September 20, 1996