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Cartesian Products of Relations and Relational Structures [1]

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Summary. In this paper the definitions of cartesian products of relations and relational structures are introduced. Facts about these notions are proved. This work is the continuation of formalization of [8].

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The articles [11], [7], [14], [16], [15], [5], [12], [10], [6], [9], [3], [13], [2], [1], [17], and [4] provide the terminology and notation for this paper.

1. Preliminaries

In this article we present several logical schemes. The scheme *FraenkelA2* concerns a non empty set $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s,t) : s$ ranges over elements of $\mathcal{A}$, t ranges over elements of $\mathcal{A}$, $\mathcal{P}[s,t]\}$ is a subset of $\mathcal{A}$

provided the following condition is met:

- For every element s of $\mathcal{A}$ and for every element t of $\mathcal{A}$ holds $\mathcal{F}(s,t) \in \mathcal{A}$.

The scheme *ExtensionalityR* deals with binary relations $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

$\mathcal{A} = \mathcal{B}$

provided the following requirements are met:

- For all sets a, b holds $\langle a, b\rangle \in \mathcal{A}$ iff $\mathcal{P}[a,b]$,
- For all sets a, b holds $\langle a, b\rangle \in \mathcal{B}$ iff $\mathcal{P}[a,b]$.

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Let X be an empty set. Observe that $\pi_1(X)$ is empty and $\pi_2(X)$ is empty.

Let X, Y be non empty sets and let D be a non empty subset of $[\!: X, Y :\!]$. Observe that $\pi_1(D)$ is non empty and $\pi_2(D)$ is non empty.

Let L be a non empty relational structure and let X be an empty subset of L. Observe that $\downarrow X$ is empty.

Let L be a non empty relational structure and let X be an empty subset of L. Observe that $\uparrow X$ is empty.

The following propositions are true:

(1) For all sets X, Y and for every subset D of $[\!: X, Y :\!]$ holds $D \subseteq [\!: \pi_1(D), \pi_2(D) :\!]$.

(2) Let L be a transitive antisymmetric relational structure with g.l.b.'s and let a, b, c, d be elements of L. If $a \leq c$ and $b \leq d$, then $a \sqcap b \leq c \sqcap d$.

(3) Let L be a transitive antisymmetric relational structure with l.u.b.'s and let a, b, c, d be elements of L. If $a \leq c$ and $b \leq d$, then $a \sqcup b \leq c \sqcup d$.

(4) Let L be a complete reflexive antisymmetric non empty relational structure, and let D be a subset of L, and let x be an element of L. If $x \in D$, then $\sup D \sqcap x = x$.

(5) Let L be a complete reflexive antisymmetric non empty relational structure, and let D be a subset of L, and let x be an element of L. If $x \in D$, then $\inf D \sqcup x = x$.

(6) For every non empty relational structure L and for all subsets X, Y of L such that $X \subseteq Y$ holds $\downarrow X \subseteq \downarrow Y$.

(7) For every non empty relational structure L and for all subsets X, Y of L such that $X \subseteq Y$ holds $\uparrow X \subseteq \uparrow Y$.

(8) Let S, T be posets with g.l.b.'s, and let f be a map from S into T, and let x, y be elements of S. Then f preserves inf of $\{x, y\}$ if and only if $f(x \sqcap y) = f(x) \sqcap f(y)$.

(9) Let S, T be posets with l.u.b.'s, and let f be a map from S into T, and let x, y be elements of S. Then f preserves sup of $\{x, y\}$ if and only if $f(x \sqcup y) = f(x) \sqcup f(y)$.

Now we present four schemes. The scheme *Inf Union* concerns a complete antisymmetric non empty relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

$\bigsqcap_{\mathcal{A}}\{\bigsqcap_{\mathcal{A}} X : X$ ranges over subsets of $\mathcal{A}$, $\mathcal{P}[X]\} \geq \bigsqcap_{\mathcal{A}} \bigcup\{X : X$ ranges over subsets of $\mathcal{A}$, $\mathcal{P}[X]\}$

for all values of the parameters.

The scheme *Inf of Infs* deals with a complete lattice $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

$\bigsqcap_{\mathcal{A}}\{\bigsqcap_{\mathcal{A}} X : X$ ranges over subsets of $\mathcal{A}$, $\mathcal{P}[X]\} = \bigsqcap_{\mathcal{A}} \bigcup\{X : X$ ranges over subsets of $\mathcal{A}$, $\mathcal{P}[X]\}$

for all values of the parameters.

The scheme *Sup Union* concerns a complete antisymmetric non empty relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}} X : X$ ranges over subsets of $\mathcal{A}$, $\mathcal{P}[X]\} \leq \bigsqcup_{\mathcal{A}} \bigcup\{X : X$ ranges over subsets of $\mathcal{A}$, $\mathcal{P}[X]\}$

for all values of the parameters.

The scheme *Sup of Sups* concerns a complete lattice $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}} X : X$ ranges over subsets of $\mathcal{A}$, $\mathcal{P}[X]\} = \bigsqcup_{\mathcal{A}} \bigcup\{X : X$ ranges over subsets of $\mathcal{A}$, $\mathcal{P}[X]\}$

for all values of the parameters.

2. Properties of Cartesian Products of Relational Structures

Let P, R be binary relations. The functor $P \times R$ yielding a binary relation is defined by:

(Def. 1) For all sets x, y holds $\langle x, y\rangle \in P \times R$ iff there exist sets p, q, s, t such that $x = \langle p, q\rangle$ and $y = \langle s, t\rangle$ and $\langle p, s\rangle \in P$ and $\langle q, t\rangle \in R$.

One can prove the following proposition

(10) Let P, R be binary relations and let x be a set. Then $x \in P \times R$ if and only if the following conditions are satisfied:

- (i) $\langle (x_{\mathbf{1}})_{\mathbf{1}}, (x_{\mathbf{2}})_{\mathbf{1}}\rangle \in P$,
- (ii) $\langle (x_{\mathbf{1}})_{\mathbf{2}}, (x_{\mathbf{2}})_{\mathbf{2}}\rangle \in R$,
- (iii) there exist sets a, b such that $x = \langle a, b\rangle$,
- (iv) there exist sets c, d such that $x_{\mathbf{1}} = \langle c, d\rangle$, and
- (v) there exist sets e, f such that $x_{\mathbf{2}} = \langle e, f\rangle$.

Let A, B, X, Y be sets, let P be a relation between A and B, and let R be a relation between X and Y. Then $P \times R$ is a relation between $[\![A, X]\!]$ and $[\![B, Y]\!]$.

Let X, Y be relational structures. The functor $[\![X, Y]\!]$ yielding a strict relational structure is defined by the conditions (Def. 2).

(Def. 2) (i) The carrier of $[\![X, Y]\!] = [\![$ the carrier of X, the carrier of $Y]\!]$, and

(ii) the internal relation of $[\![X, Y]\!] =$ (the internal relation of X)$\times$(the internal relation of Y).

Let L_1, L_2 be relational structures and let D be a subset of the carrier of $[\![L_1, L_2]\!]$. Then $\pi_1(D)$ is a subset of L_1. Then $\pi_2(D)$ is a subset of L_2.

Let S_1, S_2 be relational structures, let D_1 be a subset of the carrier of S_1, and let D_2 be a subset of the carrier of S_2. Then $[\![D_1, D_2]\!]$ is a subset of $[\![S_1, S_2]\!]$.

Let S_1, S_2 be non empty relational structures, let x be an element of the carrier of S_1, and let y be an element of the carrier of S_2. Then $\langle x, y\rangle$ is an element of $[\![S_1, S_2]\!]$.

Let L_1, L_2 be non empty relational structures and let x be an element of the carrier of $[\![L_1, L_2]\!]$. Then $x_{\mathbf{1}}$ is an element of L_1. Then $x_{\mathbf{2}}$ is an element of L_2.

The following three propositions are true:

(11) Let S_1, S_2 be non empty relational structures, and let a, c be elements of S_1, and let b, d be elements of S_2. Then $a \leq c$ and $b \leq d$ if and only if $\langle a, b\rangle \leq \langle c, d\rangle$.

(12) Let S_1, S_2 be non empty relational structures and let x, y be elements of $[: S_1, S_2 :]$. Then $x \leq y$ if and only if the following conditions are satisfied:

(i) $x_{\mathbf{1}} \leq y_{\mathbf{1}}$, and

(ii) $x_{\mathbf{2}} \leq y_{\mathbf{2}}$.

(13) Let A, B be relational structures, and let C be a non empty relational structure, and let f, g be maps from $[: A, B :]$ into C. Suppose that for every element x of A and for every element y of B holds $f(\langle x, y\rangle) = g(\langle x, y\rangle)$. Then $f = g$.

Let X, Y be non empty relational structures. Note that $[: X, Y :]$ is non empty.

Let X, Y be reflexive relational structures. Note that $[: X, Y :]$ is reflexive.

Let X, Y be antisymmetric relational structures. Note that $[: X, Y :]$ is antisymmetric.

Let X, Y be transitive relational structures. One can verify that $[: X, Y :]$ is transitive.

Let X, Y be relational structures with l.u.b.'s. One can verify that $[: X, Y :]$ has l.u.b.'s.

Let X, Y be relational structures with g.l.b.'s. One can verify that $[: X, Y :]$ has g.l.b.'s.

The following propositions are true:

(14) For all relational structures X, Y such that $[: X, Y :]$ is non empty holds X is non empty and Y is non empty.

(15) For all non empty relational structures X, Y such that $[: X, Y :]$ is reflexive holds X is reflexive and Y is reflexive.

(16) Let X, Y be non empty reflexive relational structures. If $[: X, Y :]$ is antisymmetric, then X is antisymmetric and Y is antisymmetric.

(17) Let X, Y be non empty reflexive relational structures. If $[: X, Y :]$ is transitive, then X is transitive and Y is transitive.

(18) For all non empty reflexive relational structures X, Y such that $[: X, Y :]$ has l.u.b.'s holds X has l.u.b.'s and Y has l.u.b.'s.

(19) For all non empty reflexive relational structures X, Y such that $[: X, Y :]$ has g.l.b.'s holds X has g.l.b.'s and Y has g.l.b.'s.

Let S_1, S_2 be relational structures, let D_1 be a directed subset of S_1, and let D_2 be a directed subset of S_2. Then $[: D_1, D_2 :]$ is a directed subset of $[: S_1, S_2 :]$.

We now state three propositions:

(20) Let S_1, S_2 be non empty relational structures, and let D_1 be a non empty subset of S_1, and let D_2 be a non empty subset of S_2. If $[: D_1, D_2 :]$ is directed, then D_1 is directed and D_2 is directed.

(21) For all non empty relational structures S_1, S_2 and for every non empty subset D of $[: S_1, S_2 :]$ holds $\pi_1(D)$ is non empty and $\pi_2(D)$ is non empty.

(22) Let S_1, S_2 be non empty reflexive relational structures and let D be a non empty directed subset of $[: S_1, S_2 :]$. Then $\pi_1(D)$ is directed and $\pi_2(D)$ is directed.

Let S_1, S_2 be relational structures, let D_1 be a filtered subset of S_1, and let D_2 be a filtered subset of S_2. Then $[: D_1, D_2 :]$ is a filtered subset of $[: S_1, S_2 :]$.

Next we state two propositions:

(23) Let S_1, S_2 be non empty relational structures, and let D_1 be a non empty subset of S_1, and let D_2 be a non empty subset of S_2. If $[: D_1, D_2 :]$ is filtered, then D_1 is filtered and D_2 is filtered.

(24) Let S_1, S_2 be non empty reflexive relational structures and let D be a non empty filtered subset of $[: S_1, S_2 :]$. Then $\pi_1(D)$ is filtered and $\pi_2(D)$ is filtered.

Let S_1, S_2 be relational structures, let D_1 be an upper subset of S_1, and let D_2 be an upper subset of S_2. Then $[: D_1, D_2 :]$ is an upper subset of $[: S_1, S_2 :]$.

We now state two propositions:

(25) Let S_1, S_2 be non empty reflexive relational structures, and let D_1 be a non empty subset of S_1, and let D_2 be a non empty subset of S_2. If $[: D_1, D_2 :]$ is upper, then D_1 is upper and D_2 is upper.

(26) Let S_1, S_2 be non empty reflexive relational structures and let D be a non empty upper subset of $[: S_1, S_2 :]$. Then $\pi_1(D)$ is upper and $\pi_2(D)$ is upper.

Let S_1, S_2 be relational structures, let D_1 be a lower subset of S_1, and let D_2 be a lower subset of S_2. Then $[: D_1, D_2 :]$ is a lower subset of $[: S_1, S_2 :]$.

Next we state two propositions:

(27) Let S_1, S_2 be non empty reflexive relational structures, and let D_1 be a non empty subset of S_1, and let D_2 be a non empty subset of S_2. If $[: D_1, D_2 :]$ is lower, then D_1 is lower and D_2 is lower.

(28) Let S_1, S_2 be non empty reflexive relational structures and let D be a non empty lower subset of $[: S_1, S_2 :]$. Then $\pi_1(D)$ is lower and $\pi_2(D)$ is lower.

Let R be a relational structure. We say that R is void if and only if:

(Def. 3) The internal relation of R is empty.

Let us observe that every relational structure which is empty is also void.

Let us note that there exists a poset which is non void, non empty, and strict.

One can check that every relational structure which is non void is also non empty.

Let us observe that every relational structure which is non empty and reflexive is also non void.

Let R be a non void relational structure. One can check that the internal relation of R is non empty.

Next we state a number of propositions:

(29) Let S_1, S_2 be non empty relational structures, and let D_1 be a non empty subset of S_1, and let D_2 be a non empty subset of S_2, and let x be

an element of S_1, and let y be an element of S_2. If $\langle x, y\rangle \geq [: D_1, D_2 :]$, then $x \geq D_1$ and $y \geq D_2$.

(30) Let S_1, S_2 be non empty relational structures, and let D_1 be a subset of S_1, and let D_2 be a subset of S_2, and let x be an element of S_1, and let y be an element of S_2. If $x \geq D_1$ and $y \geq D_2$, then $\langle x, y\rangle \geq [: D_1, D_2 :]$.

(31) Let S_1, S_2 be non empty relational structures, and let D be a subset of $[: S_1, S_2 :]$, and let x be an element of S_1, and let y be an element of S_2. Then $\langle x, y\rangle \geq D$ if and only if $x \geq \pi_1(D)$ and $y \geq \pi_2(D)$.

(32) Let S_1, S_2 be non empty relational structures, and let D_1 be a non empty subset of S_1, and let D_2 be a non empty subset of S_2, and let x be an element of S_1, and let y be an element of S_2. If $\langle x, y\rangle \leq [: D_1, D_2 :]$, then $x \leq D_1$ and $y \leq D_2$.

(33) Let S_1, S_2 be non empty relational structures, and let D_1 be a subset of S_1, and let D_2 be a subset of S_2, and let x be an element of S_1, and let y be an element of S_2. If $x \leq D_1$ and $y \leq D_2$, then $\langle x, y\rangle \leq [: D_1, D_2 :]$.

(34) Let S_1, S_2 be non empty relational structures, and let D be a subset of $[: S_1, S_2 :]$, and let x be an element of S_1, and let y be an element of S_2. Then $\langle x, y\rangle \leq D$ if and only if $x \leq \pi_1(D)$ and $y \leq \pi_2(D)$.

(35) Let S_1, S_2 be antisymmetric non empty relational structures, and let D_1 be a subset of S_1, and let D_2 be a subset of S_2, and let x be an element of S_1, and let y be an element of S_2. Suppose sup D_1 exists in S_1 and sup D_2 exists in S_2 and for every element b of $[: S_1, S_2 :]$ such that $b \geq [: D_1, D_2 :]$ holds $\langle x, y\rangle \leq b$. Then for every element c of S_1 such that $c \geq D_1$ holds $x \leq c$ and for every element d of S_2 such that $d \geq D_2$ holds $y \leq d$.

(36) Let S_1, S_2 be antisymmetric non empty relational structures, and let D_1 be a subset of S_1, and let D_2 be a subset of S_2, and let x be an element of S_1, and let y be an element of S_2. Suppose inf D_1 exists in S_1 and inf D_2 exists in S_2 and for every element b of $[: S_1, S_2 :]$ such that $b \leq [: D_1, D_2 :]$ holds $\langle x, y\rangle \geq b$. Then for every element c of S_1 such that $c \leq D_1$ holds $x \geq c$ and for every element d of S_2 such that $d \leq D_2$ holds $y \geq d$.

(37) Let S_1, S_2 be antisymmetric non empty relational structures, and let D_1 be a non empty subset of S_1, and let D_2 be a non empty subset of S_2, and let x be an element of S_1, and let y be an element of S_2. Suppose for every element c of S_1 such that $c \geq D_1$ holds $x \leq c$ and for every element d of S_2 such that $d \geq D_2$ holds $y \leq d$. Let b be an element of $[: S_1, S_2 :]$. If $b \geq [: D_1, D_2 :]$, then $\langle x, y\rangle \leq b$.

(38) Let S_1, S_2 be antisymmetric non empty relational structures, and let D_1 be a non empty subset of S_1, and let D_2 be a non empty subset of S_2, and let x be an element of S_1, and let y be an element of S_2. Suppose for every element c of S_1 such that $c \geq D_1$ holds $x \geq c$ and for every element d of S_2 such that $d \geq D_2$ holds $y \geq d$. Let b be an element of $[: S_1, S_2 :]$. If $b \geq [: D_1, D_2 :]$, then $\langle x, y\rangle \geq b$.

(39) Let S_1, S_2 be antisymmetric non empty relational structures, and let

D_1 be a non empty subset of S_1, and let D_2 be a non empty subset of S_2. Then sup D_1 exists in S_1 and sup D_2 exists in S_2 if and only if sup $[: D_1, D_2 :]$ exists in $[: S_1, S_2 :]$.

(40) Let S_1, S_2 be antisymmetric non empty relational structures, and let D_1 be a non empty subset of S_1, and let D_2 be a non empty subset of S_2. Then inf D_1 exists in S_1 and inf D_2 exists in S_2 if and only if inf $[: D_1, D_2 :]$ exists in $[: S_1, S_2 :]$.

(41) Let S_1, S_2 be antisymmetric non empty relational structures and let D be a subset of $[: S_1, S_2 :]$. Then sup $\pi_1(D)$ exists in S_1 and sup $\pi_2(D)$ exists in S_2 if and only if sup D exists in $[: S_1, S_2 :]$.

(42) Let S_1, S_2 be antisymmetric non empty relational structures and let D be a subset of $[: S_1, S_2 :]$. Then inf $\pi_1(D)$ exists in S_1 and inf $\pi_2(D)$ exists in S_2 if and only if inf D exists in $[: S_1, S_2 :]$.

(43) Let S_1, S_2 be antisymmetric non empty relational structures, and let D_1 be a non empty subset of S_1, and let D_2 be a non empty subset of S_2. If sup D_1 exists in S_1 and sup D_2 exists in S_2, then $\sup[: D_1, D_2 :] = \langle \sup D_1, \sup D_2 \rangle$.

(44) Let S_1, S_2 be antisymmetric non empty relational structures, and let D_1 be a non empty subset of S_1, and let D_2 be a non empty subset of S_2. If inf D_1 exists in S_1 and inf D_2 exists in S_2, then $\inf[: D_1, D_2 :] = \langle \inf D_1, \inf D_2 \rangle$.

Let X, Y be complete antisymmetric non empty relational structures. Observe that $[: X, Y :]$ is complete.

We now state several propositions:

(45) Let X, Y be non empty lower-bounded antisymmetric relational structures. If $[: X, Y :]$ is complete, then X is complete and Y is complete.

(46) Let L_1, L_2 be antisymmetric non empty relational structures and let D be a non empty subset of $[: L_1, L_2 :]$. If $[: L_1, L_2 :]$ is complete or sup D exists in $[: L_1, L_2 :]$, then $\sup D = \langle \sup \pi_1(D), \sup \pi_2(D) \rangle$.

(47) Let L_1, L_2 be antisymmetric non empty relational structures and let D be a non empty subset of $[: L_1, L_2 :]$. If $[: L_1, L_2 :]$ is complete or inf D exists in $[: L_1, L_2 :]$, then $\inf D = \langle \inf \pi_1(D), \inf \pi_2(D) \rangle$.

(48) For all non empty reflexive relational structures S_1, S_2 and for every non empty directed subset D of $[: S_1, S_2 :]$ holds $[: \pi_1(D), \pi_2(D) :] \subseteq \downarrow D$.

(49) For all non empty reflexive relational structures S_1, S_2 and for every non empty filtered subset D of $[: S_1, S_2 :]$ holds $[: \pi_1(D), \pi_2(D) :] \subseteq \uparrow D$.

The scheme *Kappa2DS* concerns non empty relational structures $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a binary functor $\mathcal{F}$ yielding a set, and states that:

There exists a map f from $[: \mathcal{A}, \mathcal{B} :]$ into $\mathcal{C}$ such that for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $f(\langle x, y \rangle) = \mathcal{F}(x, y)$

provided the following requirement is met:

- For every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\mathcal{F}(x, y)$ is an element of $\mathcal{C}$.

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