FORMALIZED MATHEMATICS
Volume 6, Number **2**, 1997
University of Białystok

Reconstructions of Special Sequences[1]

Yatsuka Nakamura
Shinshu University
Nagano

Roman Matuszewski
Warsaw University
Białystok

Summary. We discuss here some methods for reconstructing special sequences which generate special polygonal arcs in $\mathcal{E}^2_{\mathrm{T}}$. For such reconstructions we introduce a "mid" function which cuts out the middle part of a sequence; the "$\rfloor$" function, which cuts down the left part of a sequence at some point; the "$\lfloor$" function for cutting down the right part at some point; and the "$\rfloor\lfloor$" function for cutting down both sides at two given points.

We also introduce some methods glueing two special sequences. By such cutting and glueing methods, the speciality of sequences (generatability of special polygonal arcs) is shown to be preserved.

MML Identifier: `JORDAN3`.

The papers [12], [15], [14], [8], [1], [9], [10], [2], [3], [13], [4], [16], [7], [11], [6], and [5] provide the notation and terminology for this paper.

1. Preliminaries

We adopt the following convention: n, i, i_1, i_2, j denote natural numbers and D denotes a non empty set.

We now state a number of propositions:

(1) For all natural numbers i, i_1 such that $i \geqslant i_1$ or $i -' i_1 \geqslant 1$ or $i - i_1 \geqslant 1$ holds $i -' i_1 = i - i_1$.

(2) For every natural number n holds $n -' 0 = n$.

(3) For all natural numbers i_1, i_2 holds $i_1 - i_2 \leqslant i_1 -' i_2$.

(4) For all natural numbers n, i_1, i_2 such that $i_1 \leqslant i_2$ holds $n -' i_2 \leqslant n -' i_1$.

[1]The work has been done while the second author was visiting Nagano in autumn 1996.

ISSN 1426–2630

(5) For all n, i_1, i_2 such that $i_1 \leqslant i_2$ holds $i_1 -' n \leqslant i_2 -' n$.

(6) For all natural numbers i, i_1 such that $i \geqslant i_1$ or $i -' i_1 \geqslant 1$ or $i - i_1 \geqslant 1$ holds $i -' i_1 + i_1 = i$.

(7) For all natural numbers i_1, i_2 such that $i_1 \leqslant i_2$ holds $i_1 -' 1 \leqslant i_2$.

(8) For every i holds $i -' 2 = i -' 1 -' 1$.

(9) For all i_1, i_2 such that $i_1 + 1 \leqslant i_2$ holds $i_1 < i_2$ and $i_1 -' 1 < i_2$ and $i_1 -' 2 < i_2$ and $i_1 \leqslant i_2$.

(10) Let given i_1, i_2. Suppose $i_1 + 2 \leqslant i_2$ or $i_1 + 1 + 1 \leqslant i_2$. Then $i_1 + 1 < i_2$ and $(i_1+1) -' 1 < i_2$ and $(i_1+1) -' 2 < i_2$ and $i_1 + 1 \leqslant i_2$ and $i_1 -' 1 + 1 < i_2$ and $(i_1 -' 1 + 1) -' 1 < i_2$ and $i_1 < i_2$ and $i_1 -' 1 < i_2$ and $i_1 -' 2 < i_2$ and $i_1 \leqslant i_2$.

(11) For all i_1, i_2 such that $i_1 \leqslant i_2$ or $i_1 \leqslant i_2 -' 1$ holds $i_1 < i_2 + 1$ and $i_1 \leqslant i_2 + 1$ and $i_1 < i_2 + 1 + 1$ and $i_1 \leqslant i_2 + 1 + 1$ and $i_1 < i_2 + 2$ and $i_1 \leqslant i_2 + 2$.

(12) For all i_1, i_2 such that $i_1 < i_2$ or $i_1 + 1 \leqslant i_2$ holds $i_1 \leqslant i_2 -' 1$.

(13) For all i, i_1, i_2 such that $i \geqslant i_1$ holds $i \geqslant i_1 -' i_2$.

(14) For all i, i_1 such that $1 \leqslant i$ and $1 \leqslant i_1 -' i$ holds $i_1 -' i < i_1$.

(15) For all finite sequences p, q and for every i such that $\operatorname{len} p < i$ but $i \leqslant \operatorname{len} p + \operatorname{len} q$ or $i \leqslant \operatorname{len}(p \frown q)$ holds $(p \frown q)(i) = q(i - \operatorname{len} p)$.

(16) Let x be arbitrary and f be a finite sequence of elements of D. Then $\operatorname{len}(f \frown \langle x \rangle) = \operatorname{len} f + 1$ and $\operatorname{len}(\langle x \rangle \frown f) = \operatorname{len} f + 1$ and $(f \frown \langle x \rangle)(\operatorname{len} f + 1) = x$ and $(\langle x \rangle \frown f)(1) = x$.

(17) Let x be arbitrary and f be a finite sequence of elements of D. Suppose $1 \leqslant \operatorname{len} f$. Then $(f \frown \langle x \rangle)(1) = f(1)$ and $(f \frown \langle x \rangle)(1) = \pi_1 f$ and $(\langle x \rangle \frown f)(\operatorname{len} f + 1) = f(\operatorname{len} f)$ and $(\langle x \rangle \frown f)(\operatorname{len} f + 1) = \pi_{\operatorname{len} f} f$.

(18) For every finite sequence f of elements of D such that $\operatorname{len} f = 1$ holds $\operatorname{Rev}(f) = f$.

(19) For every finite sequence f of elements of D and for every natural number k holds $\operatorname{len}(f_{\lfloor k}) = \operatorname{len} f -' k$.

(20) Let f be a finite sequence of elements of D and k be a natural number. If $1 \leqslant k$ and $k \leqslant n$ and $n \leqslant \operatorname{len} f$, then $(f \restriction n)(k) = f(k)$.

(21) For every finite sequence f of elements of D and for all natural numbers l_1, l_2 holds $f_{\lfloor l_1} \restriction l_2 -' l_1 = (f \restriction l_2)_{\lfloor l_1}$.

2. Middle Function for Finite Sequences

Let us consider D, let f be a finite sequence of elements of D, and let k_1, k_2 be natural numbers. The functor $\operatorname{mid}(f, k_1, k_2)$ yields a finite sequence of elements of D and is defined by:

(Def. 1)(i) $\operatorname{mid}(f, k_1, k_2) = f_{\lfloor k_1 -' 1} \restriction (k_2 -' k_1 + 1)$ if $k_1 \leqslant k_2$,

(ii) $\operatorname{mid}(f, k_1, k_2) = \operatorname{Rev}(f_{\lfloor k_2 -' 1} \restriction (k_1 -' k_2 + 1))$, otherwise.

The following propositions are true:

(22) Let f be a finite sequence of elements of D and k_1, k_2 be natural numbers. If $1 \leqslant k_1$ and $k_1 \leqslant \operatorname{len} f$ and $1 \leqslant k_2$ and $k_2 \leqslant \operatorname{len} f$, then $\operatorname{Rev}(\operatorname{mid}(f, k_1, k_2)) = \operatorname{mid}(\operatorname{Rev}(f), \operatorname{len} f -' k_2 + 1, \operatorname{len} f -' k_1 + 1)$.

(23) Let n, m be natural numbers and f be a finite sequence of elements of D. If $1 \leqslant m$ and $m + n \leqslant \operatorname{len} f$, then $f_{\lfloor n}(m) = f(m + n)$ and $f_{\lfloor n}(m) = f(n + m)$.

(24) Let i be a natural number and f be a finite sequence of elements of D. If $1 \leqslant i$ and $i \leqslant \operatorname{len} f$, then $(\operatorname{Rev}(f))(i) = f((\operatorname{len} f - i) + 1)$.

(25) For every finite sequence f of elements of D and for every natural number k such that $1 \leqslant k$ holds $\operatorname{mid}(f, 1, k) = f \restriction k$.

(26) For every finite sequence f of elements of D and for every natural number k such that $k \leqslant \operatorname{len} f$ holds $\operatorname{mid}(f, k, \operatorname{len} f) = f_{\lfloor k -' 1}$.

(27) Let f be a finite sequence of elements of D and k_1, k_2 be natural numbers. Suppose $1 \leqslant k_1$ and $k_1 \leqslant \operatorname{len} f$ and $1 \leqslant k_2$ and $k_2 \leqslant \operatorname{len} f$. Then

(i) $(\operatorname{mid}(f, k_1, k_2))(1) = f(k_1)$,

(ii) if $k_1 \leqslant k_2$, then $\operatorname{len} \operatorname{mid}(f, k_1, k_2) = k_2 -' k_1 + 1$ and for every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{mid}(f, k_1, k_2)$ holds $(\operatorname{mid}(f, k_1, k_2))(i) = f((i + k_1) -' 1)$, and

(iii) if $k_1 > k_2$, then $\operatorname{len} \operatorname{mid}(f, k_1, k_2) = k_1 -' k_2 + 1$ and for every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{mid}(f, k_1, k_2)$ holds $(\operatorname{mid}(f, k_1, k_2))(i) = f(k_1 -' i + 1)$.

(28) For every finite sequence f of elements of D and for all natural numbers k_1, k_2 such that $1 \leqslant \operatorname{len} f$ holds $\operatorname{rng} \operatorname{mid}(f, k_1, k_2) \subseteq \operatorname{rng} f$.

(29) For every finite sequence f of elements of D such that $1 \leqslant \operatorname{len} f$ holds $\operatorname{mid}(f, 1, \operatorname{len} f) = f$.

(30) For every finite sequence f of elements of D such that $1 \leqslant \operatorname{len} f$ holds $\operatorname{mid}(f, \operatorname{len} f, 1) = \operatorname{Rev}(f)$.

(31) Let f be a finite sequence of elements of D and k_1, k_2, i be natural numbers. Suppose $1 \leqslant k_1$ and $k_1 \leqslant k_2$ and $k_2 \leqslant \operatorname{len} f$ and $1 \leqslant i$ and $i \leqslant k_2 -' k_1 + 1$ or $i \leqslant (k_2 - k_1) + 1$ or $i \leqslant (k_2 + 1) - k_1$. Then $(\operatorname{mid}(f, k_1, k_2))(i) = f((i + k_1) -' 1)$ and $(\operatorname{mid}(f, k_1, k_2))(i) = f(i -' 1 + k_1)$ and $(\operatorname{mid}(f, k_1, k_2))(i) = f((i + k_1) - 1)$ and $(\operatorname{mid}(f, k_1, k_2))(i) = f((i - 1) + k_1)$.

(32) Let f be a finite sequence of elements of D and k, i be natural numbers. If $1 \leqslant i$ and $i \leqslant k$ and $k \leqslant \operatorname{len} f$, then $(\operatorname{mid}(f, 1, k))(i) = f(i)$.

(33) Let f be a finite sequence of elements of D and k_1, k_2 be natural numbers. If $1 \leqslant k_1$ and $k_1 \leqslant k_2$ and $k_2 \leqslant \operatorname{len} f$, then $\operatorname{len} \operatorname{mid}(f, k_1, k_2) \leqslant \operatorname{len} f$.

(34) For every finite sequence f of elements of $\mathcal{E}^n_{\mathrm{T}}$ such that $2 \leqslant \operatorname{len} f$ holds $f(1) \in \widetilde{\mathcal{L}}(f)$ and $\pi_1 f \in \widetilde{\mathcal{L}}(f)$ and $f(\operatorname{len} f) \in \widetilde{\mathcal{L}}(f)$ and $\pi_{\operatorname{len} f} f \in \widetilde{\mathcal{L}}(f)$.

(35) For every finite sequence f of elements of $\mathcal{E}^n_{\mathrm{T}}$ and for every natural number i holds $\mathcal{L}(f, i) \subseteq \widetilde{\mathcal{L}}(f)$.

(36) For every finite sequence f of elements of $\mathcal{E}^n_{\mathrm{T}}$ such that $\mathrm{len}\, f \geqslant 2$ holds $f(1) \in \widetilde{\mathcal{L}}(f)$ and $\pi_1 f \in \widetilde{\mathcal{L}}(f)$ and $f(\mathrm{len}\, f) \in \widetilde{\mathcal{L}}(f)$ and $\pi_{\mathrm{len}\, f} f \in \widetilde{\mathcal{L}}(f)$.

(37) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ but $q_1 \in \mathcal{L}(p_1, p_2)$ but $q_2 \in \mathcal{L}(p_1, p_2)$ holds $(q_1)_{\mathbf{1}} = (q_2)_{\mathbf{1}}$ or $(q_1)_{\mathbf{2}} = (q_2)_{\mathbf{2}}$.

(38) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ but $\mathcal{L}(q_1, q_2) \subseteq \mathcal{L}(p_1, p_2)$ holds $(q_1)_{\mathbf{1}} = (q_2)_{\mathbf{1}}$ or $(q_1)_{\mathbf{2}} = (q_2)_{\mathbf{2}}$.

(39) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. If $2 \leqslant n$ and f is a special sequence, then $f \restriction n$ is a special sequence.

(40) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Suppose $n \leqslant \mathrm{len}\, f$ and $2 \leqslant \mathrm{len}\, f -' n$ and f is a special sequence. Then $f_{\lfloor n}$ is a special sequence.

(41) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and k_1, k_2 be natural numbers. Suppose f is a special sequence and $1 \leqslant k_1$ and $k_1 \leqslant \mathrm{len}\, f$ and $1 \leqslant k_2$ and $k_2 \leqslant \mathrm{len}\, f$ and $k_1 \neq k_2$. Then $\mathrm{mid}(f, k_1, k_2)$ is a special sequence.

3. A Concept of Index for Finite Sequences in $\mathcal{E}^2_{\mathrm{T}}$

Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Let us assume that f is a special sequence and there exists a natural number i such that $1 \leqslant i$ and $i + 1 \leqslant \mathrm{len}\, f$ and $p \in \mathcal{L}(f, i)$. The functor $\mathrm{Index}(p, f)$ yielding a natural number is defined as follows:

(Def. 2) $1 \leqslant \mathrm{Index}(p, f)$ and $\mathrm{Index}(p, f) + 1 \leqslant \mathrm{len}\, f$ and $p \in \mathcal{L}(f, \mathrm{Index}(p, f))$ and $p \neq f(\mathrm{Index}(p, f) + 1)$ or $\mathrm{Index}(p, f) = \mathrm{len}\, f$ and $p = f(\mathrm{len}\, f)$.

One can prove the following propositions:

(42) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$. Then $1 \leqslant \mathrm{Index}(p, f)$ and $\mathrm{Index}(p, f)+1 \leqslant \mathrm{len}\, f$ and $p \in \mathcal{L}(f, \mathrm{Index}(p, f))$ and $p \neq f(\mathrm{Index}(p, f)+1)$ or $\mathrm{Index}(p, f) = \mathrm{len}\, f$ and $p = f(\mathrm{len}\, f)$.

(43) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and there exists a natural number i such that $1 \leqslant i$ and $i + 1 \leqslant \mathrm{len}\, f$ and $p \in \mathcal{L}(f, i)$. Then $1 \leqslant \mathrm{Index}(p, f)$ and $\mathrm{Index}(p, f) \leqslant \mathrm{len}\, f$.

(44) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and there exists a natural number i such that $1 \leqslant i$ and $i + 1 \leqslant \mathrm{len}\, f$ and $p \in \mathcal{L}(f, i)$ and $p \neq f(\mathrm{len}\, f)$. Then $\mathrm{Index}(p, f) < \mathrm{len}\, f$.

(45) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and given i_1. Suppose that

(i) f is a special sequence,

(ii) there exists a natural number i such that $1 \leqslant i$ and $i + 1 \leqslant \mathrm{len}\, f$ and $p \in \mathcal{L}(f, i)$, and

(iii) $1 \leqslant i_1$ and $i_1+1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i_1)$ and $p \neq f(i_1+1)$ or $i_1 = \operatorname{len} f$ and $p = f(\operatorname{len} f)$.
Then $i_1 = \operatorname{Index}(p, f)$.

(46) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and there exists a natural number i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i)$ and $p = f(\operatorname{len} f)$. Then $\operatorname{Index}(p, f) = \operatorname{len} f$.

(47) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and given i_1. If f is a special sequence and $1 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} f$ and $p = f(i_1)$, then $\operatorname{Index}(p, f) = i_1$.

(48) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and given i_1. Suppose f is a special sequence and $1 \leqslant i_1$ and $i_1 + 1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i_1)$. Then $i_1 = \operatorname{Index}(p, f)$ or $i_1 + 1 = \operatorname{Index}(p, f)$.

Let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. We say that g is a special sequence joining p_1, p_2 if and only if:

(Def. 3) g is a special sequence and $g(1) = p_1$ and $g(\operatorname{len} g) = p_2$.

One can prove the following propositions:

(49) Let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $1 \leqslant \operatorname{len} g$ and g is a special sequence joining p_1, p_2. Then $\operatorname{Rev}(g)$ is a special sequence joining p_2, p_1.

(50) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and given j. Suppose that
(i) f is a special sequence,
(ii) there exists a natural number i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i)$,
(iii) $p \neq f(\operatorname{len} f)$,
(iv) $g = \langle p \rangle \frown \operatorname{mid}(f, \operatorname{Index}(p, f) + 1, \operatorname{len} f)$,
(v) $1 \leqslant j$, and
(vi) $j + 1 \leqslant \operatorname{len} g$.
Then $\mathcal{L}(g, j) \subseteq \mathcal{L}(f, (\operatorname{Index}(p, f) + j) -' 1)$.

(51) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that
(i) f is a special sequence,
(ii) there exists a natural number i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i)$,
(iii) $p \neq f(\operatorname{len} f)$, and
(iv) $g = \langle p \rangle \frown \operatorname{mid}(f, \operatorname{Index}(p, f) + 1, \operatorname{len} f)$.
Then g is a special sequence joining p, $\pi_{\operatorname{len} f} f$.

(52) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and given j. Suppose that
(i) f is a special sequence,
(ii) there exists a natural number i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i)$,

(iii) $1 \leqslant j$,
(iv) $j + 1 \leqslant \operatorname{len} g$,
(v) if $p \neq f(\operatorname{Index}(p, f))$, then $g = (\operatorname{mid}(f, 1, \operatorname{Index}(p, f))) \frown \langle p \rangle$, and
(vi) if $p = f(\operatorname{Index}(p, f))$, then $g = \operatorname{mid}(f, 1, \operatorname{Index}(p, f))$.
Then $\mathcal{L}(g, j) \subseteq \mathcal{L}(f, j)$.

(53) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that
(i) f is a special sequence,
(ii) there exists a natural number i such that $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} f$ and $p \in \mathcal{L}(f, i)$,
(iii) $p \neq f(1)$,
(iv) if $p \neq f(\operatorname{Index}(p, f))$, then $g = (\operatorname{mid}(f, 1, \operatorname{Index}(p, f))) \frown \langle p \rangle$, and
(v) if $p = f(\operatorname{Index}(p, f))$, then $g = \operatorname{mid}(f, 1, \operatorname{Index}(p, f))$.
Then g is a special sequence joining $\pi_1 f$, p.

4. Left and Right Cutting Functions for Finite Sequences in $\mathcal{E}^2_{\mathrm{T}}$

Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\downharpoonright p, f$ yielding a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 4) $\downharpoonright p, f = \langle p \rangle \frown \operatorname{mid}(f, \operatorname{Index}(p, f) + 1, \operatorname{len} f)$.

The functor $\downharpoonleft f, p$ yields a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 5)(i) $\downharpoonleft f, p = (\operatorname{mid}(f, 1, \operatorname{Index}(p, f))) \frown \langle p \rangle$ if $p \neq f(\operatorname{Index}(p, f))$,
(ii) $\downharpoonleft f, p = \operatorname{mid}(f, 1, \operatorname{Index}(p, f))$, otherwise.

Next we state four propositions:

(54) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\operatorname{len} f)$. Then $(\downharpoonright p, f)(1) = p$ and for every i such that $1 < i$ and $i \leqslant (\operatorname{len} f - \operatorname{Index}(p, f)) + 1$ holds $(\downharpoonright p, f)(i) = f((\operatorname{Index}(p, f) + i) - 1)$.

(55) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $(\downharpoonleft f, p)(\operatorname{len} \downharpoonleft f, p) = p$ and for every i such that $1 < i$ and $i \leqslant \operatorname{Index}(p, f)$ holds $(\downharpoonleft f, p)(i) = f(i)$.

(56) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\operatorname{len} f)$, then $\operatorname{len} \downharpoonright p, f = (\operatorname{len} f - \operatorname{Index}(p, f)) + 1$.

(57) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$ such that f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$. Then
(i) if $p \neq f(\operatorname{Index}(p, f))$, then $\operatorname{len} \downharpoonleft f, p = \operatorname{Index}(p, f) + 1$, and
(ii) if $p = f(\operatorname{Index}(p, f))$, then $\operatorname{len} \downharpoonleft f, p = \operatorname{Index}(p, f)$.

Let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. The predicate $\mathrm{LE}(q_1, q_2, p_1, p_2)$ is defined by the conditions (Def. 6).

(Def. 6)(i) $q_1 \in \mathcal{L}(p_1, p_2)$,

(ii) $q_2 \in \mathcal{L}(p_1, p_2)$, and

(iii) for all real numbers r_1, r_2 such that $0 \leqslant r_1$ and $r_1 \leqslant 1$ and $q_1 = (1 - r_1) \cdot p_1 + r_1 \cdot p_2$ and $0 \leqslant r_2$ and $r_2 \leqslant 1$ and $q_2 = (1 - r_2) \cdot p_1 + r_2 \cdot p_2$ holds $r_1 \leqslant r_2$.

Let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. The predicate $\mathrm{LT}(q_1, q_2, p_1, p_2)$ is defined as follows:

(Def. 7) $\mathrm{LE}(q_1, q_2, p_1, p_2)$ and $q_1 \neq q_2$.

Next we state several propositions:

(58) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{LT}(q_1, q_2, p_1, p_2)$ holds $\mathrm{LE}(q_1, q_2, p_1, p_2)$.

(59) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{LE}(q_1, q_2, p_1, p_2)$ and $\mathrm{LE}(q_2, q_1, p_1, p_2)$ holds $q_1 = q_2$.

(60) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $q_1 \in \mathcal{L}(p_1, p_2)$ and $q_2 \in \mathcal{L}(p_1, p_2)$ and $p_1 \neq p_2$ holds $\mathrm{LE}(q_1, q_2, p_1, p_2)$ or $\mathrm{LT}(q_2, q_1, p_1, p_2)$ but $\mathrm{LE}(q_1, q_2, p_1, p_2)$ but $\mathrm{LT}(q_2, q_1, p_1, p_2)$.

(61) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q, p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $\mathrm{Index}(p, f) < \mathrm{Index}(q, f)$, then $q \in \widetilde{\mathcal{L}}(\rfloor p, f)$.

(62) For all points p, q, p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{LE}(p, q, p_1, p_2)$ holds $q \in \mathcal{L}(p, p_2)$ and $p \in \mathcal{L}(p_1, q)$.

(63) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q, p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$ and $\mathrm{Index}(p, f) = \mathrm{Index}(q, f)$ and $\mathrm{LE}(p, q, \pi_{\mathrm{Index}(p,f)} f, \pi_{\mathrm{Index}(p,f)+1} f)$. Then $q \in \widetilde{\mathcal{L}}(\rfloor p, f)$.

5. Cutting Both Sides of a Finite Sequence and a Discussion of Speciality of Sequences in $\mathcal{E}^2_{\mathrm{T}}$

Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\rfloor\lfloor p, f, q$ yielding a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 8)(i) $\rfloor\lfloor p, f, q = \lfloor\rfloor p, f, q$ if $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $\mathrm{Index}(p, f) < \mathrm{Index}(q, f)$ or $\mathrm{Index}(p, f) = \mathrm{Index}(q, f)$ and $\mathrm{LE}(p, q, \pi_{\mathrm{Index}(p,f)} f, \pi_{\mathrm{Index}(p,f)+1} f)$,

(ii) $\rfloor\lfloor p, f, q = \mathrm{Rev}(\lfloor\rfloor q, f, p)$, otherwise.

The following propositions are true:

(64) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$. Then $\rfloor p, f$ is a special sequence joining p, $\pi_{\mathrm{len}\, f} f$.

(65) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$. Then $\rfloor p, f$ is a special sequence.

(66) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $\lfloor f, p$ is a special sequence joining $\pi_1 f$, p.

(67) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $\lfloor f, p$ is a special sequence.

(68) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p, q be points of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$. Then $\rfloor \lfloor p, f, q$ is a special sequence joining p, q.

(69) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p, q be points of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$. Then $\rfloor \lfloor p, f, q$ is a special sequence.

(70) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $f(\mathrm{len}\, f) = g(1)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $f \frown \mathrm{mid}(g, 2, \mathrm{len}\, g)$ is a special sequence.

(71) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $f(\mathrm{len}\, f) = g(1)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $f \frown \mathrm{mid}(g, 2, \mathrm{len}\, g)$ is a special sequence joining $\pi_1 f$, $\pi_{\mathrm{len}\, g} g$.

(72) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for every natural number n holds $\widetilde{\mathcal{L}}(f_{\restriction n}) \subseteq \widetilde{\mathcal{L}}(f)$.

(73) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$ and f is a special sequence, then $\widetilde{\mathcal{L}}(\rfloor p, f) \subseteq \widetilde{\mathcal{L}}(f)$.

(74) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\mathrm{len}\, f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq f(\mathrm{len}\, f)$. Then $(\rfloor p, f) \frown \mathrm{mid}(g, 2, \mathrm{len}\, g)$ is a special sequence joining p, $\pi_{\mathrm{len}\, g} g$.

(75) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\mathrm{len}\, f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq f(\mathrm{len}\, f)$. Then $(\rfloor p, f) \frown \mathrm{mid}(g, 2, \mathrm{len}\, g)$ is a special sequence.

(76) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $f(\mathrm{len}\, f) = g(1)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $(\mathrm{mid}(f, 1, \mathrm{len}\, f -' 1)) \frown g$ is a special sequence.

(77) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $f(\mathrm{len}\, f) = g(1)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $(\mathrm{mid}(f, 1, \mathrm{len}\, f -' 1)) \frown g$ is a special sequence joining $\pi_1 f$, $\pi_{\mathrm{len}\, g} g$.

(78) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$ and f is a special sequence, then $\widetilde{\mathcal{L}}(\downharpoonleft f, p) \subseteq \widetilde{\mathcal{L}}(f)$.

(79) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(g)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq g(1)$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown \downharpoonleft g, p$ is a special sequence joining $\pi_1 f$, p.

(80) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(g)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq g(1)$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown \downharpoonleft g, p$ is a special sequence.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Some properties of restrictions of finite sequences. *Formalized Mathematics*, 5(**2**):241–245, 1996.

[5] Czesław Byliński and Yatsuka Nakamura. Special polygons. *Formalized Mathematics*, 5(**2**):247–252, 1996.

[6] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[7] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\rm t}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[9] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Formalized Mathematics*, 3(**2**):275–278, 1992.

[10] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.

[11] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[13] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[14] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[15] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[16] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received December 10, 1996