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Closure Operators and Subalgebras[1]

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The notation and terminology used in this paper are introduced in the following papers: [19], [22], [11], [23], [24], [9], [10], [1], [4], [18], [15], [17], [20], [2], [21], [3], [16], [13], [5], [6], [14], [25], [12], [8], and [7].

1. Preliminaries

In this article we present several logical schemes. The scheme *SubrelstrEx* concerns a non empty relational structure $\mathcal{A}$, a set $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

> There exists a non empty full strict relational substructure S of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds x is an element of S if and only if $\mathcal{P}[x]$

provided the following conditions are met:

- $\mathcal{P}[\mathcal{B}]$,
- $\mathcal{B} \in$ the carrier of $\mathcal{A}$.

The scheme *RelstrEq* deals with non empty relational structures $\mathcal{A}$, $\mathcal{B}$, a unary predicate $\mathcal{P}$, and a binary predicate $\mathcal{Q}$, and states that:

> The relational structure of $\mathcal{A}$ = the relational structure of $\mathcal{B}$

provided the following conditions are met:

- For every set x holds x is an element of $\mathcal{A}$ iff $\mathcal{P}[x]$,
- For every set x holds x is an element of $\mathcal{B}$ iff $\mathcal{P}[x]$,
- For all elements a, b of $\mathcal{A}$ holds $a \leqslant b$ iff $\mathcal{Q}[a, b]$,
- For all elements a, b of $\mathcal{B}$ holds $a \leqslant b$ iff $\mathcal{Q}[a, b]$.

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The scheme *SubrelstrEq1* deals with a non empty relational structure $\mathcal{A}$, non empty full relational substructures $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

The relational structure of $\mathcal{B}$ = the relational structure of $\mathcal{C}$

provided the following conditions are satisfied:

- For every set x holds x is an element of $\mathcal{B}$ iff $\mathcal{P}[x]$,
- For every set x holds x is an element of $\mathcal{C}$ iff $\mathcal{P}[x]$.

The scheme *SubrelstrEq2* concerns a non empty relational structure $\mathcal{A}$, non empty full relational substructures $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

The relational structure of $\mathcal{B}$ = the relational structure of $\mathcal{C}$

provided the parameters have the following properties:

- For every element x of $\mathcal{A}$ holds x is an element of $\mathcal{B}$ iff $\mathcal{P}[x]$,
- For every element x of $\mathcal{A}$ holds x is an element of $\mathcal{C}$ iff $\mathcal{P}[x]$.

The following four propositions are true:

(1) For all binary relations R, Q holds $R \subseteq Q$ iff $R^{\smile} \subseteq Q^{\smile}$ and $R^{\smile} \subseteq Q$ iff $R \subseteq Q^{\smile}$.

(2) For every binary relation R and for every set X holds $(R|^2 X)^{\smile} = R^{\smile}|^2 X$.

(3) Let L, S be relational structures. Then

(i) S is a relational substructure of L iff $S^{\rm op}$ is a relational substructure of $L^{\rm op}$, and

(ii) $S^{\rm op}$ is a relational substructure of L iff S is a relational substructure of $L^{\rm op}$.

(4) Let L, S be relational structures. Then

(i) S is a full relational substructure of L iff $S^{\rm op}$ is a full relational substructure of $L^{\rm op}$, and

(ii) $S^{\rm op}$ is a full relational substructure of L iff S is a full relational substructure of $L^{\rm op}$.

Let L be a relational structure and let S be a full relational substructure of L. Then $S^{\rm op}$ is a strict full relational substructure of $L^{\rm op}$.

Let X be a set and let L be a non empty relational structure. Observe that $X \longmapsto L$ is nonempty.

Let S be a relational structure and let T be a non empty reflexive relational structure. One can verify that there exists a map from S into T which is monotone.

Let L be a non empty relational structure. One can check that every map from L into L which is projection is also monotone and idempotent.

Let S, T be non empty reflexive relational structures and let f be a monotone map from S into T. One can verify that f° is monotone.

Let L be a 1-sorted structure. Note that id_L is one-to-one.

Let L be a non empty reflexive relational structure. One can check that id_L is sups-preserving and infs-preserving.

The following proposition is true

(5) Let L be a relational structure and S be a subset of L. Then id_S is a map from $\mathrm{sub}(S)$ into L and for every map f from $\mathrm{sub}(S)$ into L such that $f = \mathrm{id}_S$ holds f is monotone.

Let L be a non empty reflexive relational structure. Note that there exists a map from L into L which is sups-preserving, infs-preserving, closure, kernel, and one-to-one.

One can prove the following proposition

(6) Let L be a non empty reflexive relational structure, c be a closure map from L into L, and x be an element of L. Then $c(x) \geqslant x$.

Let S, T be 1-sorted structures, let f be a function from the carrier of S into the carrier of T, and let R be a 1-sorted structure. Let us assume that the carrier of $R \subseteq$ the carrier of S. The functor $f{\restriction}R$ yields a map from R into T and is defined by:

(Def. 1) $f{\restriction}R = f{\restriction}$the carrier of R.

One can prove the following propositions:

(7) Let S, T be relational structures, R be a relational substructure of S, and f be a function from the carrier of S into the carrier of T. Then $f{\restriction}R = f{\restriction}$the carrier of R and for every set x such that $x \in$ the carrier of R holds $(f{\restriction}R)(x) = f(x)$.

(8) Let S, T be relational structures and f be a map from S into T. Suppose f is one-to-one. Let R be a relational substructure of S. Then $f{\restriction}R$ is one-to-one.

Let S, T be non empty reflexive relational structures, let f be a monotone map from S into T, and let R be a relational substructure of S. Note that $f{\restriction}R$ is monotone.

One can prove the following proposition

(9) Let S, T be non empty relational structures, R be a non empty relational substructure of S, f be a map from S into T, and g be a map from T into S. Suppose f is one-to-one and $g = f^{-1}$. Then $g{\restriction}\mathrm{Im}(f{\restriction}R)$ is a map from $\mathrm{Im}(f{\restriction}R)$ into R and $g{\restriction}\mathrm{Im}(f{\restriction}R) = (f{\restriction}R)^{-1}$.

2. The lattice of closure operators

Let S be a relational structure and let T be a non empty reflexive relational structure. Note that $\mathrm{MonMaps}(S,T)$ is non empty.

Next we state the proposition

(10) Let S be a relational structure, T be a non empty reflexive relational structure, and x be a set. Then x is an element of $\mathrm{MonMaps}(S,T)$ if and only if x is a monotone map from S into T.

Let L be a non empty reflexive relational structure. The functor $\mathrm{ClOpers}(L)$ yields a non empty full strict relational substructure of $\mathrm{MonMaps}(L,L)$ and is defined by:

(Def. 2) For every map f from L into L holds f is an element of ClOpers(L) iff f is closure.

The following propositions are true:

(11) Let L be a non empty reflexive relational structure and x be a set. Then x is an element of ClOpers(L) if and only if x is a closure map from L into L.

(12) Let X be a set, L be a non empty relational structure, f, g be functions from X into the carrier of L, and x, y be elements of L^X. If $x = f$ and $y = g$, then $x \leqslant y$ iff $f \leqslant g$.

(13) Let L be a complete lattice, c_1, c_2 be maps from L into L, and x, y be elements of ClOpers(L). If $x = c_1$ and $y = c_2$, then $x \leqslant y$ iff $c_1 \leqslant c_2$.

(14) Let L be a reflexive relational structure and S_1, S_2 be full relational substructures of L. Suppose the carrier of $S_1 \subseteq$ the carrier of S_2. Then S_1 is a relational substructure of S_2.

(15) Let L be a complete lattice and c_1, c_2 be closure maps from L into L. Then $c_1 \leqslant c_2$ if and only if $\mathrm{Im}\, c_2$ is a relational substructure of $\mathrm{Im}\, c_1$.

3. The lattice of closure systems

Let L be a relational structure. The functor Sub(L) yields a strict non empty relational structure and is defined by the conditions (Def. 3).

(Def. 3)(i) For every set x holds x is an element of Sub(L) iff x is a strict relational substructure of L, and

(ii) for all elements a, b of Sub(L) holds $a \leqslant b$ iff there exists a relational structure R such that $b = R$ and a is a relational substructure of R.

One can prove the following proposition

(16) Let L, R be relational structures and x, y be elements of Sub(L). Suppose $y = R$. Then $x \leqslant y$ if and only if x is a relational substructure of R.

Let L be a relational structure. One can verify that Sub(L) is reflexive antisymmetric and transitive.

Let L be a relational structure. Observe that Sub(L) is complete.

Let L be a complete lattice. Note that every relational substructure of L which is infs-inheriting is also non empty and every relational substructure of L which is sups-inheriting is also non empty.

Let L be a relational structure. A system of L is a full relational substructure of L.

Let L be a non empty relational structure and let S be a system of L. We introduce S is closure as a synonym of S is infs-inheriting.

Let L be a non empty relational structure. Observe that $\mathrm{sub}(\Omega_L)$ is infs-inheriting and sups-inheriting.

Let L be a non empty relational structure. The functor ClosureSystems(L) yields a full strict non empty relational substructure of Sub(L) and is defined by the condition (Def. 4).

(Def. 4) Let R be a strict relational substructure of L. Then R is an element of ClosureSystems(L) if and only if R is infs-inheriting and full.

Next we state two propositions:

(17) Let L be a non empty relational structure and x be a set. Then x is an element of ClosureSystems(L) if and only if x is a strict closure system of L.

(18) Let L be a non empty relational structure, R be a relational structure, and x, y be elements of ClosureSystems(L). Suppose $y = R$. Then $x \leqslant y$ if and only if x is a relational substructure of R.

4. Isomorphism between closure operators and closure systems

Let L be a non empty poset and let h be a closure map from L into L. Note that $\mathrm{Im}\, h$ is infs-inheriting.

Let L be a non empty poset. The functor ClImageMap(L) yields a map from ClOpers(L) into $(\mathrm{ClosureSystems}(L))^{\mathrm{op}}$ and is defined as follows:

(Def. 5) For every closure map c from L into L holds $(\mathrm{ClImageMap}(L))(c) = \mathrm{Im}\, c$.

Let L be a non empty relational structure and let S be a relational substructure of L. The closure operation of S is a map from L into L and is defined by:

(Def. 6) For every element x of L holds (the closure operation of S)(x) $= \bigsqcap_L(\uparrow x \cap$ the carrier of S).

Let L be a complete lattice and let S be a closure system of L. One can verify that the closure operation of S is closure.

Next we state two propositions:

(19) Let L be a complete lattice and S be a closure system of L. Then Im (the closure operation of S) = the relational structure of S.

(20) For every complete lattice L and for every closure map c from L into L holds the closure operation of $\mathrm{Im}\, c = c$.

Let L be a complete lattice. One can check that ClImageMap(L) is one-to-one.

One can prove the following propositions:

(21) For every complete lattice L holds $(\mathrm{ClImageMap}(L))^{-1}$ is a map from $(\mathrm{ClosureSystems}(L))^{\mathrm{op}}$ into ClOpers(L).

(22) Let L be a complete lattice and S be a strict closure system of L. Then $(\mathrm{ClImageMap}(L))^{-1}(S)$ = the closure operation of S.

Let L be a complete lattice. One can verify that ClImageMap(L) is isomorphic.

The following proposition is true

(23) For every complete lattice L holds $\mathrm{ClOpers}(L)$ and $(\mathrm{ClosureSystems}(L))^{\mathrm{op}}$ are isomorphic.

5. Isomorphism between closure operators preserving directed sups and subalgebras

We now state three propositions:

(24) Let L be a relational structure, S be a full relational substructure of L, and X be a subset of S. Then

(i) if X is a directed subset of L, then X is directed, and

(ii) if X is a filtered subset of L, then X is filtered.

(25) Let L be a complete lattice and S be a closure system of L. Then the closure operation of S is directed-sups-preserving if and only if S is directed-sups-inheriting.

(26) Let L be a complete lattice and h be a closure map from L into L. Then h is directed-sups-preserving if and only if $\mathrm{Im}\, h$ is directed-sups-inheriting.

Let L be a complete lattice and let S be a directed-sups-inheriting closure system of L. Observe that the closure operation of S is directed-sups-preserving.

Let L be a complete lattice and let h be a directed-sups-preserving closure map from L into L. Observe that $\mathrm{Im}\, h$ is directed-sups-inheriting.

Let L be a non empty reflexive relational structure. The functor $\mathrm{ClOpers}^*(L)$ yields a non empty full strict relational substructure of $\mathrm{ClOpers}(L)$ and is defined by the condition (Def. 7).

(Def. 7) Let f be a closure map from L into L. Then f is an element of $\mathrm{ClOpers}^*(L)$ if and only if f is directed-sups-preserving.

Next we state the proposition

(27) Let L be a non empty reflexive relational structure and x be a set. Then x is an element of $\mathrm{ClOpers}^*(L)$ if and only if x is a directed-sups-preserving closure map from L into L.

Let L be a non empty relational structure. The functor $\mathrm{Subalgebras}(L)$ yields a full strict non empty relational substructure of $\mathrm{ClosureSystems}(L)$ and is defined by the condition (Def. 8).

(Def. 8) Let R be a strict closure system of L. Then R is an element of $\mathrm{Subalgebras}(L)$ if and only if R is directed-sups-inheriting.

The following two propositions are true:

(28) Let L be a non empty relational structure and x be a set. Then x is an element of $\mathrm{Subalgebras}(L)$ if and only if x is a strict directed-sups-inheriting closure system of L.

(29) For every complete lattice L holds $\mathrm{Im}(\mathrm{ClImageMap}(L) \restriction \mathrm{ClOpers}^*(L)) = (\mathrm{Subalgebras}(L))^{\mathrm{op}}$.

Let L be a complete lattice. Note that $(\mathrm{ClImageMap}(L) \restriction \mathrm{ClOpers}^*(L))^\circ$ is isomorphic.

The following proposition is true

(30) For every complete lattice L holds $\mathrm{ClOpers}^*(L)$ and $(\mathrm{Subalgebras}(L))^{\mathrm{op}}$ are isomorphic.

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