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Scott Topology[1]

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Summary. In the article we continue the formalization in Mizar of [15, 98–105]. We work with structures of the form

$$L = \langle C, \leqslant, \tau \rangle,$$

where C is the carrier of the structure, $\leqslant$ - an ordering relation on C and τ a family of subsets of C. When $\langle C, \leqslant \rangle$ is a complete lattice we say that L is Scott, if τ is the Scott topology of $\langle C, \leqslant \rangle$. We define the Scott convergence (lim inf convergence). Following [15] we prove that in the case of a continuous lattice $\langle C, \leqslant \rangle$ the Scott convergence is topological, i.e. enjoys the properties: (CONSTANTS), (SUBNETS), (DIVERGENCE), (ITERATED LIMITS). We formalize the theorem, that if the Scott convergence has the (ITERATED LIMITS) property, the $\langle C, \leqslant \rangle$ is continuous.

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The terminology and notation used in this paper are introduced in the following articles: [29], [35], [37], [25], [12], [14], [36], [10], [11], [9], [3], [8], [33], [23], [27], [38], [28], [26], [41], [17], [30], [2], [24], [1], [22], [34], [4], [5], [6], [16], [40], [13], [18], [19], [20], [7], [39], [32], [21], and [31].

1. Preliminaries

The scheme *Irrel* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, a binary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

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$\{\mathcal{F}(u), u$ ranges over elements of $\mathcal{A} : \mathcal{P}[u]\} = \{\mathcal{F}(i, v), i$ ranges over elements of $\mathcal{B}, v$ ranges over elements of $\mathcal{A} : \mathcal{P}[v]\}$

provided the following condition is met:

- For every element i of $\mathcal{B}$ and for every element u of $\mathcal{A}$ holds $\mathcal{F}(u) = \mathcal{F}(i, u)$.

One can prove the following three propositions:

(1) Let L be a complete non empty lattice and X, Y be subsets of the carrier of L. If Y is coarser than X, then $\bigsqcap_L X \leqslant \bigsqcap_L Y$.

(2) Let L be a complete non empty lattice and X, Y be subsets of the carrier of L. If X is finer than Y then $\bigsqcup_L X \leqslant \bigsqcup_L Y$.

(3) Let T be a relational structure, A be an upper subset of T, and B be a directed subset of T. Then $A \cap B$ is directed.

Let T be a reflexive non empty relational structure. Observe that there exists a subset of T which is non empty, directed, and finite.

Next we state the proposition

(4) For every non empty poset T with l.u.b.'s and for every non empty directed finite subset D of T holds $\sup D \in D$.

Let us observe that there exists a relational structure which is trivial, reflexive, transitive, non empty, antisymmetric, finite, and strict and has l.u.b.'s.

Let us observe that there exists a 1-sorted structure which is finite, non empty, and strict.

Let T be a finite 1-sorted structure. Note that every subset of T is finite.

Let R be a relational structure. Note that $\emptyset_R$ is lower and upper.

Let R be a trivial non empty relational structure. Note that every subset of R is upper.

One can prove the following propositions:

(5) Let T be a non empty relational structure, x be an element of T, and A be an upper subset of T. If $x \notin A$, then A misses $\downarrow x$.

(6) Let T be a non empty relational structure, x be an element of T, and A be a lower subset of T. If $x \in A$, then $\downarrow x \subseteq A$.

2. Scott Topology

Let T be a non empty reflexive relational structure and let S be a subset of T. We say that S is inaccessible by directed joins if and only if:

(Def. 1) For every non empty directed subset D of T such that $\sup D \in S$ holds D meets S.

We introduce S is inaccessible as a synonym of S is inaccessible by directed joins. We say that S is closed under directed sups if and only if:

(Def. 2) For every non empty directed subset D of T such that $D \subseteq S$ holds $\sup D \in S$.

We introduce S is directly closed as a synonym of S is closed under directed sups. We say that S is property(S) if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let D be a non empty directed subset of T. Suppose $\sup D \in S$. Then there exists an element y of T such that $y \in D$ and for every element x of T such that $x \in D$ and $x \geqslant y$ holds $x \in S$.

We introduce S has the property (S) as a synonym of S is property(S).

Let T be a non empty reflexive relational structure. One can check that $\emptyset_T$ is property(S) and directly closed.

Let T be a non empty reflexive relational structure. Observe that there exists a subset of T which is property(S) and directly closed.

Let T be a non empty reflexive relational structure and let S be a property(S) subset of T. One can verify that $-S$ is directly closed.

Let T be a reflexive non empty FR-structure. We say that T is Scott if and only if:

(Def. 4) For every subset S of T holds S is open iff S is inaccessible and upper.

Let T be a reflexive transitive antisymmetric non empty finite relational structure with l.u.b.'s. Note that every subset of T is inaccessible.

Let T be a reflexive transitive antisymmetric non empty finite FR-structure with l.u.b.'s. Let us observe that T is Scott if and only if:

(Def. 5) For every subset S of T holds S is open iff S is upper.

Let us mention that there exists a non empty strict TopLattice which is trivial, complete, and Scott.

Let T be a non empty reflexive relational structure. Observe that Ω_T is directly closed and inaccessible.

Let T be a non empty reflexive relational structure. Note that there exists a subset of T which is directly closed, lower, inaccessible, and upper.

Let T be a complete non empty TopLattice and let S be an inaccessible subset of T. Note that $-S$ is directly closed.

Let T be a non empty reflexive relational structure and let S be a directly closed subset of T. One can check that $-S$ is inaccessible.

One can prove the following propositions:

(7) Let T be a complete Scott non empty TopLattice and S be a subset of T. Then S is closed if and only if S is directly closed and lower.

(8) For every complete non empty TopLattice T and for every element x of T holds $\downarrow x$ is directly closed.

(9) For every complete Scott non empty TopLattice T and for every element x of T holds $\overline{\{x\}} = \downarrow x$.

(10) Every complete Scott non empty TopLattice is a T_0-space.

(11) For every complete Scott non empty TopLattice T and for every element x of T holds $\downarrow x$ is closed.

(12) For every complete Scott non empty TopLattice T and for every element x of T holds $-\downarrow x$ is open.

(13) Let T be a complete Scott non empty TopLattice, x be an element of T, and A be an upper subset of T. If $x \notin A$, then $-{\downarrow}x$ is a neighbourhood of A.

(14) Let T be a complete Scott non empty TopLattice and S be an upper subset of T. Then there exists a family F of subsets of T such that $S = \bigcap F$ and for every subset X of T such that $X \in F$ holds X is a neighbourhood of S.

(15) Let T be a Scott non empty TopLattice and S be a subset of T. Then S is open if and only if S is upper and property(S).

Let T be a complete non empty TopLattice. Observe that every subset of T which is lower is also property(S).

One can prove the following proposition

(16) Let T be a non empty transitive reflexive FR-structure. Suppose the topology of $T = \{S, S$ ranges over subsets of T: S has the property (S)$\}$. Then T is topological space-like.

3. Scott Convergence

In the sequel R will be a non empty relational structure, N will be a net in R, and i, j will be elements of the carrier of N.

Let us consider R, N. The functor $\liminf N$ yielding an element of R is defined by:

(Def. 6) $\liminf N = \bigsqcup_R \{\bigsqcap_R \{N(i) : i \geqslant j\} : j$ ranges over elements of the carrier of $N\}$.

Let R be a reflexive non empty relational structure, let N be a net in R, and let p be an element of the carrier of R. We say that p is S-limit of N if and only if:

(Def. 7) $p \leqslant \liminf N$.

Let R be a reflexive non empty relational structure. The Scott convergence of R yields a convergence class of R and is defined by the condition (Def. 8).

(Def. 8) Let N be a strict net in R. Suppose $N \in \mathrm{NetUniv}(R)$. Let p be an element of the carrier of R. Then $\langle N, p\rangle \in$ the Scott convergence of R if and only if p is S-limit of N.

The following two propositions are true:

(17) Let R be a non empty complete lattice, N be a net in R, and p, q be elements of the carrier of R. If p is S-limit of N and N is eventually in ${\downarrow}q$, then $p \leqslant q$.

(18) Let R be a non empty complete lattice, N be a net in R, and p, q be elements of the carrier of R. If N is eventually in ${\uparrow}q$, then $\liminf N \geqslant q$.

Let R be a reflexive non empty relational structure and let N be a non empty net structure over R. Let us observe that N is monotone if and only if:

(Def. 9) For all elements i, j of the carrier of N such that $i \leqslant j$ holds $N(i) \leqslant N(j)$.

Let R be a non empty relational structure, let S be a non empty set, and let f be a function from S into the carrier of R. The functor $\mathrm{NetStr}(S, f)$ yielding a strict non empty net structure over R is defined by the conditions (Def. 10).

(Def. 10)(i) The carrier of $\mathrm{NetStr}(S, f) = S$,

(ii) the mapping of $\mathrm{NetStr}(S, f) = f$, and

(iii) for all elements i, j of $\mathrm{NetStr}(S, f)$ holds $i \leqslant j$ iff $(\mathrm{NetStr}(S, f))(i) \leqslant (\mathrm{NetStr}(S, f))(j)$.

The following two propositions are true:

(19) Let L be a non empty 1-sorted structure and N be a non empty net structure over L. Then rng (the mapping of N) $= \{N(i) : i$ ranges over elements of the carrier of $N\}$.

(20) Let R be a non empty relational structure, S be a non empty set, and f be a function from S into the carrier of R. If $\mathrm{rng}\, f$ is directed, then $\mathrm{NetStr}(S, f)$ is directed.

Let R be a non empty relational structure, let S be a non empty set, and let f be a function from S into the carrier of R. Note that $\mathrm{NetStr}(S, f)$ is monotone.

Let R be a transitive non empty relational structure, let S be a non empty set, and let f be a function from S into the carrier of R. Note that $\mathrm{NetStr}(S, f)$ is transitive.

Let R be a reflexive non empty relational structure, let S be a non empty set, and let f be a function from S into the carrier of R. Observe that $\mathrm{NetStr}(S, f)$ is reflexive.

We now state the proposition

(21) Let R be a non empty transitive relational structure, S be a non empty set, and f be a function from S into the carrier of R. If $S \subseteq$ the carrier of R and $\mathrm{NetStr}(S, f)$ is directed, then $\mathrm{NetStr}(S, f) \in \mathrm{NetUniv}(R)$.

Let R be a non empty lattice. One can check that there exists a net in R which is monotone, reflexive, and strict.

The following propositions are true:

(22) For every non empty complete lattice R and for every monotone reflexive net N in R holds $\liminf N = \sup N$.

(23) For every complete non empty lattice R and for every constant net N in R holds the value of $N = \liminf N$.

(24) For every complete non empty lattice R and for every constant net N in R holds the value of N is S-limit of N.

Let S be a non empty 1-sorted structure and let e be an element of the carrier of S. The functor $\mathrm{NetStr}(e)$ yielding a strict net structure over S is defined as follows:

(Def. 11) The carrier of $\mathrm{NetStr}(e) = \{e\}$ and the internal relation of $\mathrm{NetStr}(e) = \{\langle e, e\rangle\}$ and the mapping of $\mathrm{NetStr}(e) = \mathrm{id}_{\{e\}}$.

Let S be a non empty 1-sorted structure and let e be an element of the carrier of S. Observe that $\mathrm{NetStr}(e)$ is non empty.

One can prove the following propositions:

(25) Let S be a non empty 1-sorted structure, e be an element of the carrier of S, and x be an element of NetStr(e). Then $x = e$.

(26) Let S be a non empty 1-sorted structure, e be an element of the carrier of S, and x be an element of NetStr(e). Then $(\mathrm{NetStr}(e))(x) = e$.

Let S be a non empty 1-sorted structure and let e be an element of the carrier of S. Observe that NetStr(e) is reflexive transitive directed and antisymmetric.

We now state several propositions:

(27) Let S be a non empty 1-sorted structure, e be an element of the carrier of S, and X be a set. Then NetStr(e) is eventually in X if and only if $e \in X$.

(28) Let S be a reflexive antisymmetric non empty relational structure and e be an element of the carrier of S. Then $e = \liminf \mathrm{NetStr}(e)$.

(29) For every non empty reflexive relational structure S and for every element e of the carrier of S holds $\mathrm{NetStr}(e) \in \mathrm{NetUniv}(S)$.

(30) Let R be a non empty complete lattice, Z be a net in R, and D be a subset of R. Suppose $D = \{\bigsqcap_R\{Z(k), k$ ranges over elements of the carrier of Z: $k \geqslant j\}$: j ranges over elements of the carrier of $Z\}$. Then D is non empty and directed.

(31) Let L be a non empty complete lattice and S be a subset of L. Then $S \in$ the topology of ConvergenceSpace(the Scott convergence of L) if and only if S is inaccessible and upper.

(32) Let T be a non empty complete Scott TopLattice. Then the topological structure of $T =$ ConvergenceSpace(the Scott convergence of T).

(33) Let T be a non empty complete TopLattice. Suppose the topological structure of $T =$ ConvergenceSpace(the Scott convergence of T). Let S be a subset of T. Then S is open if and only if S is inaccessible and upper.

(34) Let T be a non empty complete TopLattice. Suppose the topological structure of $T =$ ConvergenceSpace(the Scott convergence of T). Then T is Scott.

Let R be a complete non empty lattice. Note that the Scott convergence of R has (CONSTANTS) property.

Let R be a complete non empty lattice. Observe that the Scott convergence of R has (SUBNETS) property.

The following proposition is true

(35) Let S be a non empty 1-sorted structure, N be a net in S, X be a set, and M be a subnet of N. If $M = N^{-1}(X)$, then for every element i of the carrier of M holds $M(i) \in X$.

Let L be a non empty complete lattice. The functor sigma L yielding a family of subsets of L is defined as follows:

(Def. 12) sigma $L =$ the topology of ConvergenceSpace(the Scott convergence of L).

One can prove the following propositions:

(36) For every continuous complete Scott TopLattice L and for every element x of L holds $\uparrow x$ is open.

(37) For every non empty complete TopLattice T such that the topology of $T = \operatorname{sigma} T$ holds T is Scott.

Let R be a continuous non empty complete lattice. Observe that the Scott convergence of R is topological.

We now state a number of propositions:

(38) Let T be a continuous non empty complete Scott TopLattice, x be an element of the carrier of T, and N be a net in T. If $N \in \operatorname{NetUniv}(T)$, then x is S-limit of N iff $x \in \operatorname{Lim} N$.

(39) Let L be a complete non empty poset. Suppose the Scott convergence of L has (ITERATED LIMITS) property. Then L is continuous.

(40) Let T be a complete Scott non empty TopLattice. Then T is continuous if and only if $\operatorname{Convergence}(T) =$ the Scott convergence of T.

(41)[2] For every complete Scott non empty TopLattice T and for every upper subset S of T such that S is open holds S is open.

(42) Let L be a non empty relational structure, S be an upper subset of L, and x be an element of L. If $x \in S$, then $\uparrow x \subseteq S$.

(43) Let L be a non empty complete continuous Scott TopLattice, p be an element of L, and S be a subset of L. If S is open and $p \in S$, then there exists an element q of L such that $q \ll p$ and $q \in S$.

(44) Let L be a non empty complete continuous Scott TopLattice and p be an element of L. Then $\{\Uparrow q, q$ ranges over elements of L: $q \ll p\}$ is a basis of p.

(45) For every complete continuous Scott non empty TopLattice T holds $\{\Uparrow x :$ x ranges over elements of $T\}$ is a basis of T.

(46)[3] Let T be a complete continuous Scott non empty TopLattice and S be an upper subset of T. Then S is open if and only if S is open.

(47) For every complete continuous Scott non empty TopLattice T and for every element p of T holds $\operatorname{Int}\uparrow p = \Uparrow p$.

(48) Let T be a complete continuous Scott non empty TopLattice and S be a subset of T. Then $\operatorname{Int} S = \bigcup\{\Uparrow x, x$ ranges over elements of T: $\Uparrow x \subseteq S\}$.

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