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On the Baire Category Theorem[1]

Artur Korniłowicz
Warsaw University
Białystok

Summary. In this paper Exercise 3.43 from Chapter 1 of [14] is solved.

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The terminology and notation used in this paper have been introduced in the following articles: [23], [27], [2], [28], [10], [11], [8], [13], [25], [9], [1], [4], [21], [26], [29], [12], [17], [22], [3], [5], [16], [6], [30], [18], [19], [7], [15], [20], and [24].

1. Preliminaries

Let T be a topological structure and let A be a subset of the carrier of T. Then $\operatorname{Int} A$ is a subset of T.

Let T be a topological structure and let P be a subset of the carrier of T. Let us observe that P is closed if and only if:

(Def. 1) $-P$ is open.

Let T be a non empty topological space and let F be a family of subsets of T. We say that F is dense if and only if:

(Def. 2) For every subset X of T such that $X \in F$ holds X is dense.

The following proposition is true

(1) Let L be a non empty 1-sorted structure, A be a subset of L, and x be an element of L. Then $x \in -A$ if and only if $x \notin A$.

Let us observe that there exists a 1-sorted structure which is empty.

Let S be an empty 1-sorted structure. Note that the carrier of S is empty.

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Let S be an empty 1-sorted structure. Note that every subset of S is empty.

One can check that every set which is finite is also countable.

Let us note that there exists a set which is empty.

Let S be a 1-sorted structure. One can verify that there exists a subset of S which is empty.

One can verify that there exists a set which is non empty and finite.

Let L be a non empty relational structure. Observe that there exists a subset of L which is non empty and finite.

Let us note that $\mathbb{N}$ is infinite.

Let us note that there exists a set which is infinite and countable.

Let S be a 1-sorted structure. One can verify that there exists a family of subsets of S which is empty.

One can prove the following propositions:

(2) For all sets X, Y such that $\overline{\overline{X}} \leqslant \overline{\overline{Y}}$ and Y is countable holds X is countable.

(3) For every infinite countable set A holds $\mathbb{N} \approx A$.

(4) For every non empty countable set A there exists a function f from $\mathbb{N}$ into A such that $\operatorname{rng} f = A$.

(5) For every 1-sorted structure S and for all subsets X, Y of S holds $-(X \cup Y) = (-X) \cap -Y$.

(6) For every 1-sorted structure S and for all subsets X, Y of S holds $-X \cap Y = -X \cup -Y$.

(7) Let L be a non empty transitive relational structure and A, B be subsets of L. If A is finer than B, then $\downarrow A \subseteq \downarrow B$.

(8) Let L be a non empty transitive relational structure and A, B be subsets of L. If A is coarser than B, then $\uparrow A \subseteq \uparrow B$.

(9) Let L be a non empty poset and D be a non empty finite filtered subset of L. If $\inf D$ exists in L, then $\inf D \in D$.

(10) Let L be a lower-bounded antisymmetric non empty relational structure and X be a non empty lower subset of L. Then $\bot_L \in X$.

(11) Let L be a lower-bounded antisymmetric non empty relational structure and X be a non empty subset of L. Then $\bot_L \in \downarrow X$.

(12) Let L be an upper-bounded antisymmetric non empty relational structure and X be a non empty upper subset of L. Then $\top_L \in X$.

(13) Let L be an upper-bounded antisymmetric non empty relational structure and X be a non empty subset of L. Then $\top_L \in \uparrow X$.

(14) Let L be a lower-bounded antisymmetric relational structure with g.l.b.'s and X be a subset of L. Then $X \sqcap \{\bot_L\} \subseteq \{\bot_L\}$.

(15) Let L be a lower-bounded antisymmetric relational structure with g.l.b.'s and X be a non empty subset of L. Then $X \sqcap \{\bot_L\} = \{\bot_L\}$.

(16) Let L be an upper-bounded antisymmetric relational structure with l.u.b.'s and X be a subset of L. Then $X \sqcup \{\top_L\} \subseteq \{\top_L\}$.

(17) Let L be an upper-bounded antisymmetric relational structure with l.u.b.'s and X be a non empty subset of L. Then $X \sqcup \{\top_L\} = \{\top_L\}$.

(18) For every upper-bounded semilattice L and for every subset X of L holds $\{\top_L\} \sqcap X = X$.

(19) For every lower-bounded poset L with l.u.b.'s and for every subset X of L holds $\{\bot_L\} \sqcup X = X$.

(20) Let L be a non empty reflexive relational structure and A, B be subsets of L. If $A \subseteq B$, then A is finer than B and coarser than B.

(21) Let L be an antisymmetric transitive relational structure with g.l.b.'s, V be a subset of L, and x, y be elements of L. If $x \leqslant y$, then $\{y\} \sqcap V$ is coarser than $\{x\} \sqcap V$.

(22) Let L be an antisymmetric transitive relational structure with l.u.b.'s, V be a subset of L, and x, y be elements of L. If $x \leqslant y$, then $\{x\} \sqcup V$ is finer than $\{y\} \sqcup V$.

(23) Let L be a non empty relational structure and V, S, T be subsets of L. If S is coarser than T and V is upper and $T \subseteq V$, then $S \subseteq V$.

(24) Let L be a non empty relational structure and V, S, T be subsets of L. If S is finer than T and V is lower and $T \subseteq V$, then $S \subseteq V$.

(25) For every semilattice L and for every upper filtered subset F of L holds $F \sqcap F = F$.

(26) For every sup-semilattice L and for every lower directed subset I of L holds $I \sqcup I = I$.

(27) For every upper-bounded semilattice L and for every subset V of L holds $\{x, x$ ranges over elements of L: $V \sqcap \{x\} \subseteq V\}$ is non empty.

(28) Let L be an antisymmetric transitive relational structure with g.l.b.'s and V be a subset of L. Then $\{x, x$ ranges over elements of L: $V \sqcap \{x\} \subseteq V\}$ is a filtered subset of L.

(29) Let L be an antisymmetric transitive relational structure with g.l.b.'s and V be an upper subset of L. Then $\{x, x$ ranges over elements of L: $V \sqcap \{x\} \subseteq V\}$ is an upper subset of L.

(30) For every poset L with g.l.b.'s and for every subset X of L such that X is open and lower holds X is filtered.

Let L be a poset with g.l.b.'s. Observe that every subset of L which is open and lower is also filtered.

Let L be a continuous antisymmetric non empty reflexive relational structure. One can verify that every subset of L which is lower is also open.

Let L be a continuous semilattice and let x be an element of L. Note that $-\downarrow x$ is open.

We now state two propositions:

(31) Let L be a semilattice and C be a non empty subset of L. Suppose that for all elements x, y of L such that $x \in C$ and $y \in C$ holds $x \leqslant y$ or $y \leqslant x$. Let Y be a non empty finite subset of C. Then $\bigsqcap_L Y \in Y$.

(32) Let L be a sup-semilattice and C be a non empty subset of L. Suppose that for all elements x, y of L such that $x \in C$ and $y \in C$ holds $x \leqslant y$ or $y \leqslant x$. Let Y be a non empty finite subset of C. Then $\bigsqcup_L Y \in Y$.

Let L be a semilattice and let F be a filter of L. A subset of L is called a generator set of F if:

(Def. 3) $F = \uparrow\mathrm{fininfs}(\mathrm{it})$.

Let L be a semilattice and let F be a filter of L. One can verify that there exists a generator set of F which is non empty.

The following propositions are true:

(33) Let L be a semilattice, A be a subset of L, and B be a non empty subset of L. If A is coarser than B, then $\mathrm{fininfs}(A)$ is coarser than $\mathrm{fininfs}(B)$.

(34) Let L be a semilattice, F be a filter of L, G be a generator set of F, and A be a non empty subset of L. Suppose G is coarser than A and A is coarser than F. Then A is a generator set of F.

(35) Let L be a semilattice, A be a subset of L, and f, g be functions from $\mathbb{N}$ into the carrier of L. Suppose $\mathrm{rng}\, f = A$ and for every element n of $\mathbb{N}$ holds $g(n) = \bigsqcap_L \{f(m), m$ ranges over natural numbers: $m \leqslant n\}$. Then A is coarser than $\mathrm{rng}\, g$.

(36) Let L be a semilattice, F be a filter of L, G be a generator set of F, and f, g be functions from $\mathbb{N}$ into the carrier of L. Suppose $\mathrm{rng}\, f = G$ and for every element n of $\mathbb{N}$ holds $g(n) = \bigsqcap_L \{f(m), m$ ranges over natural numbers: $m \leqslant n\}$. Then $\mathrm{rng}\, g$ is a generator set of F.

2. On the Baire Category Theorem

The following propositions are true:

(37) Let L be a lower-bounded continuous lattice, V be an open upper subset of L, F be a filter of L, and v be an element of L. Suppose $V \sqcap F \subseteq V$ and $v \in V$ and there exists a non empty generator set of F which is countable. Then there exists an open filter O of L such that $O \subseteq V$ and $v \in O$ and $F \subseteq O$.

(38) Let L be a lower-bounded continuous lattice, V be an open upper subset of L, N be a non empty countable subset of L, and v be an element of L. Suppose $V \sqcap N \subseteq V$ and $v \in V$. Then there exists an open filter O of L such that $\{v\} \sqcap N \subseteq O$ and $O \subseteq V$ and $v \in O$.

(39) Let L be a lower-bounded continuous lattice, V be an open upper subset of L, N be a non empty countable subset of L, and x, y be elements of L. Suppose $V \sqcap N \subseteq V$ and $y \in V$ and $x \notin V$. Then there exists an irreducible element p of L such that $x \leqslant p$ and $p \notin \uparrow(\{y\} \sqcap N)$.

(40) Let L be a lower-bounded continuous lattice, x be an element of L, and N be a non empty countable subset of L. Suppose that for all elements n, y of L such that $y \nleqslant x$ and $n \in N$ holds $y \sqcap n \nleqslant x$. Let y be an element

of L. Suppose $y \not\leqslant x$. Then there exists an irreducible element p of L such that $x \leqslant p$ and $p \notin \uparrow(\{y\} \sqcap N)$.

Let L be a non empty relational structure and let u be an element of L. We say that u is dense if and only if:

(Def. 4) For every element v of L such that $v \neq \bot_L$ holds $u \sqcap v \neq \bot_L$.

Let L be an upper-bounded semilattice. Note that $\top_L$ is dense.

Let L be an upper-bounded semilattice. Note that there exists an element of L which is dense.

The following proposition is true

(41) For every non trivial bounded semilattice L and for every element x of L such that x is dense holds $x \neq \bot_L$.

Let L be a non empty relational structure and let D be a subset of L. We say that D is dense if and only if:

(Def. 5) For every element d of L such that $d \in D$ holds d is dense.

We now state the proposition

(42) For every upper-bounded semilattice L holds $\{\top_L\}$ is dense.

Let L be an upper-bounded semilattice. Note that there exists a subset of L which is non empty, finite, countable, and dense.

Next we state several propositions:

(43) Let L be a lower-bounded continuous lattice, D be a non empty countable dense subset of L, and u be an element of L. Suppose $u \neq \bot_L$. Then there exists an irreducible element p of L such that $p \neq \top_L$ and $p \notin \uparrow(\{u\} \sqcap D)$.

(44) Let T be a non empty topological space, A be an element of $\langle$the topology of T, $\subseteq\rangle$, and B be a subset of T. If $A = B$ and $-B$ is irreducible, then A is irreducible.

(45) Let T be a non empty topological space, A be an element of $\langle$the topology of T, $\subseteq\rangle$, and B be a subset of T. Suppose $A = B$ and $A \neq \top_{\langle \text{the topology of } T,\, \subseteq \rangle}$. Then A is irreducible if and only if $-B$ is irreducible.

(46) Let T be a non empty topological space, A be an element of $\langle$the topology of T, $\subseteq\rangle$, and B be a subset of T. If $A = B$, then A is dense iff B is everywhere dense.

(47) Let T be a non empty topological space. Suppose T is locally-compact. Let D be a countable family of subsets of T. Suppose D is non empty, dense, and open. Let O be a non empty subset of T. Suppose O is open. Then there exists an irreducible subset A of T such that for every subset V of T if $V \in D$, then $A \cap O \cap V \neq \emptyset$.

Let T be a non empty topological space. Let us observe that T is Baire if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let F be a family of subsets of T. Suppose F is countable and for every subset S of T such that $S \in F$ holds S is open and dense. Then Intersect(F) is dense.

Next we state the proposition

(48) For every non empty topological space T such that T is sober and locally-compact holds T is Baire.

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