FORMALIZED MATHEMATICS
Volume 6, Number 2, 1997
University of Białystok

Auxiliary and Approximating Relations[1]

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Summary. The aim of this paper is to formalize the second part of Chapter I Section 1 (1.9–1.19) in [12]. Definitions of Scott's auxiliary and approximating relations are introduced in this work. We showed that in a meet-continuous lattice, the way-below relation is the intersection of all approximating auxiliary relations (proposition (40) — compare 1.13 in [12, pp. 43–47]). By (41) a continuous lattice is a complete lattice in which $\ll$ is the smallest approximating auxiliary relation. The notions of the strong interpolation property and the interpolation property are also introduced.

MML Identifier: `WAYBEL_4`.

The articles [21], [25], [19], [10], [23], [24], [20], [9], [3], [26], [28], [7], [8], [27], [2], [4], [22], [18], [1], [17], [13], [29], [14], [15], [5], [11], [16], and [6] provide the notation and terminology for this paper.

1. Auxiliary Relations

Let L be a 1-sorted structure.

(Def. 1) A binary relation on the carrier of L is called a binary relation on L.

Let L be a non empty reflexive relational structure. The functor $\ll_L$ yields a binary relation on L and is defined as follows:

(Def. 2) For all elements x, y of L holds $\langle x, y\rangle \in \ll_L$ iff $x \ll y$.

Let L be a relational structure. The functor $\leqslant_L$ yielding a binary relation on L is defined by:

[1]This work was partially supported by the Office of Naval Research Grant N00014-95-1-1336.

ISSN 1426–2630

(Def. 3) $\leqslant_L$ = the internal relation of L.

Let L be a relational structure and let R be a binary relation on L. We say that R is auxiliary(i) if and only if:

(Def. 4) For all elements x, y of L such that $\langle x, y\rangle \in R$ holds $x \leqslant y$.

We say that R is auxiliary(ii) if and only if:

(Def. 5) For all elements x, y, z, u of L such that $u \leqslant x$ and $\langle x, y\rangle \in R$ and $y \leqslant z$ holds $\langle u, z\rangle \in R$.

Let L be a non empty relational structure and let R be a binary relation on L. We say that R is auxiliary(iii) if and only if:

(Def. 6) For all elements x, y, z of L such that $\langle x, z\rangle \in R$ and $\langle y, z\rangle \in R$ holds $\langle x \sqcup y, z\rangle \in R$.

We say that R is auxiliary(iv) if and only if:

(Def. 7) For every element x of L holds $\langle \bot_L, x\rangle \in R$.

Let L be a non empty relational structure and let R be a binary relation on L. We say that R is auxiliary if and only if:

(Def. 8) R is auxiliary(i), auxiliary(ii), auxiliary(iii), and auxiliary(iv).

Let L be a non empty relational structure. Note that every binary relation on L which is auxiliary is also auxiliary(i), auxiliary(ii), auxiliary(iii), and auxiliary(iv) and every binary relation on L which is auxiliary(i), auxiliary(ii), auxiliary(iii), and auxiliary(iv) is also auxiliary.

Let L be a lower-bounded transitive antisymmetric relational structure with l.u.b.'s. Note that there exists a binary relation on L which is auxiliary.

Next we state the proposition

(1) Let L be a lower-bounded sup-semilattice, A_1 be an auxiliary binary relation on L, and x, y, z, u be elements of L. If $\langle x, z\rangle \in A_1$ and $\langle y, u\rangle \in A_1$, then $\langle x \sqcup y, z \sqcup u\rangle \in A_1$.

Let L be a lower-bounded sup-semilattice. Observe that every binary relation on L which is auxiliary is also transitive.

Let L be a relational structure. Note that $\leqslant_L$ is auxiliary(i).

Let L be a transitive relational structure. One can verify that $\leqslant_L$ is auxiliary(ii).

Let L be an antisymmetric relational structure with l.u.b.'s. One can check that $\leqslant_L$ is auxiliary(iii).

Let L be a lower-bounded antisymmetric non empty relational structure. Note that $\leqslant_L$ is auxiliary(iv).

In the sequel a will denote a set.

Let L be a lower-bounded sup-semilattice. The functor $\mathrm{Aux}(L)$ is defined as follows:

(Def. 9) $a \in \mathrm{Aux}(L)$ iff a is an auxiliary binary relation on L.

Let L be a lower-bounded sup-semilattice. Note that $\mathrm{Aux}(L)$ is non empty.

The following two propositions are true:

(2) For every lower-bounded sup-semilattice L and for every auxiliary binary relation A_1 on L holds $A_1 \subseteq \leqslant_L$.

(3) For every lower-bounded sup-semilattice L holds $\top_{\langle \mathrm{Aux}(L), \subseteq \rangle} = \leqslant_L$.

Let L be a lower-bounded sup-semilattice. Note that $\langle \mathrm{Aux}(L), \subseteq \rangle$ is upper-bounded.

Let L be a non empty relational structure. The functor AuxBottom(L) yields a binary relation on L and is defined as follows:

(Def. 10) For all elements x, y of L holds $\langle x, y \rangle \in \mathrm{AuxBottom}(L)$ iff $x = \bot_L$.

Let L be a lower-bounded sup-semilattice. Observe that AuxBottom(L) is auxiliary.

The following propositions are true:

(4) For every lower-bounded sup-semilattice L and for every auxiliary binary relation A_1 on L holds $\mathrm{AuxBottom}(L) \subseteq A_1$.

(5) For every lower-bounded sup-semilattice L and for every auxiliary binary relation A_1 on L holds $\bot_{\langle \mathrm{Aux}(L), \subseteq \rangle} = \mathrm{AuxBottom}(L)$.

Let L be a lower-bounded sup-semilattice. One can verify that $\langle \mathrm{Aux}(L), \subseteq \rangle$ is lower-bounded.

The following two propositions are true:

(6) Let L be a lower-bounded sup-semilattice and a, b be auxiliary binary relations on L. Then $a \cap b$ is an auxiliary binary relation on L.

(7) Let L be a lower-bounded sup-semilattice and X be a non empty subset of $\langle \mathrm{Aux}(L), \subseteq \rangle$. Then $\bigcap X$ is an auxiliary binary relation on L.

Let L be a lower-bounded sup-semilattice. Note that $\langle \mathrm{Aux}(L), \subseteq \rangle$ has g.l.b.'s.

Let L be a lower-bounded sup-semilattice. Observe that $\langle \mathrm{Aux}(L), \subseteq \rangle$ is complete.

Let L be a non empty relational structure, let x be an element of L, and let A_1 be a binary relation on L. The functor $\downarrow_{A_1} x$ yields a subset of L and is defined by:

(Def. 11) $\downarrow_{A_1} x = \{y, y$ ranges over elements of L: $\langle y, x \rangle \in A_1\}$.

The functor $\uparrow_{A_1} x$ yielding a subset of L is defined by:

(Def. 12) $\uparrow_{A_1} x = \{y, y$ ranges over elements of L: $\langle x, y \rangle \in A_1\}$.

One can prove the following proposition

(8) Let L be a lower-bounded sup-semilattice, x be an element of L, and A_1 be an auxiliary(i) binary relation on L. Then $\downarrow_{A_1} x \subseteq \downarrow x$.

Let L be a lower-bounded sup-semilattice, let x be an element of L, and let A_1 be an auxiliary(ii) auxiliary(iii) auxiliary(iv) binary relation on L. Observe that $\downarrow_{A_1} x$ is directed lower and non empty.

Let L be a lower-bounded sup-semilattice and let A_1 be an auxiliary(ii) auxiliary(iii) auxiliary(iv) binary relation on L. The functor $\downarrow A_1$ yields a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ and is defined by:

(Def. 13) For every element x of L holds $(\downarrow A_1)(x) = \downarrow_{A_1} x$.

We now state three propositions:

(9) Let L be a non empty relational structure, A_1 be a binary relation on L, a be a set, and y be an element of L. Then $a \in \downarrow_{A_1} y$ if and only if $\langle a, y \rangle \in A_1$.

(10) Let L be a sup-semilattice, A_1 be a binary relation on L, and y be an element of L. Then $a \in \uparrow_{A_1} y$ if and only if $\langle y, a \rangle \in A_1$.

(11) Let L be a lower-bounded sup-semilattice, A_1 be an auxiliary binary relation on L, and x be an element of L. If $A_1 =$ the internal relation of L, then $\downarrow_{A_1} x = \downarrow x$.

Let L be a non empty poset. The functor MonSet(L) yields a strict relational structure and is defined by the conditions (Def. 14).

(Def. 14)(i) $a \in$ the carrier of MonSet(L) iff there exists a map s from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ such that $a = s$ and s is monotone and for every element x of L holds $s(x) \subseteq \downarrow x$, and

(ii) for all sets c, d holds $\langle c, d \rangle \in$ the internal relation of MonSet(L) iff there exist maps f, g from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ such that $c = f$ and $d = g$ and $c \in$ the carrier of MonSet(L) and $d \in$ the carrier of MonSet(L) and $f \leqslant g$.

One can prove the following propositions:

(12) Let L be a lower-bounded sup-semilattice. Then MonSet(L) is a full relational substructure of $(\langle \mathrm{Ids}(L), \subseteq \rangle)^{\text{the carrier of } L}$.

(13) Let L be a lower-bounded sup-semilattice, A_1 be an auxiliary binary relation on L, and x, y be elements of L. If $x \leqslant y$, then $\downarrow_{A_1} x \subseteq \downarrow_{A_1} y$.

Let L be a lower-bounded sup-semilattice and let A_1 be an auxiliary binary relation on L. Note that $\downarrow A_1$ is monotone.

Next we state the proposition

(14) Let L be a lower-bounded sup-semilattice and A_1 be an auxiliary binary relation on L. Then $\downarrow A_1 \in$ the carrier of MonSet(L).

Let L be a lower-bounded sup-semilattice. Observe that MonSet(L) is non empty.

Next we state several propositions:

(15) For every lower-bounded sup-semilattice L holds IdsMap(L) $\in$ the carrier of MonSet(L).

(16) For every lower-bounded sup-semilattice L and for every auxiliary binary relation A_1 on L holds $\downarrow A_1 \leqslant$ IdsMap(L).

(17) For every lower-bounded non empty poset L and for every ideal I of L holds $\perp_L \in I$.

(18) For every upper-bounded non empty poset L and for every filter F of L holds $\top_L \in F$.

(19) For every lower-bounded non empty poset L holds $\downarrow(\perp_L) = \{\perp_L\}$.

(20) For every upper-bounded non empty poset L holds $\uparrow(\top_L) = \{\top_L\}$.

In the sequel L is a lower-bounded sup-semilattice, A_1 is an auxiliary binary relation on L, and x is an element of L.

The following propositions are true:

(21) The carrier of $L \longmapsto \{\bot_L\}$ is a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$.

(22) The carrier of $L \longmapsto \{\bot_L\} \in$ the carrier of MonSet(L).

(23) $\langle$the carrier of $L \longmapsto \{\bot_L\}$, $\downarrow A_1\rangle \in$ the internal relation of MonSet(L).

Let us consider L. Note that MonSet(L) is upper-bounded.

Let us consider L. The functor Rel2Map(L) yields a map from $\langle \mathrm{Aux}(L), \subseteq \rangle$ into MonSet(L) and is defined by:

(Def. 15) For every A_1 holds $(\mathrm{Rel2Map}(L))(A_1) = \downarrow A_1$.

The following propositions are true:

(24) For all auxiliary binary relations R_1, R_2 on L such that $R_1 \subseteq R_2$ holds $\downarrow R_1 \leqslant \downarrow R_2$.

(25) For all auxiliary binary relations R_1, R_2 on L such that $R_1 \subseteq R_2$ holds $\downarrow_{R_1} x \subseteq \downarrow_{R_2} x$.

Let us consider L. One can verify that Rel2Map(L) is monotone.

Let us consider L. The functor Map2Rel(L) yields a map from MonSet(L) into $\langle \mathrm{Aux}(L), \subseteq \rangle$ and is defined by the condition (Def. 16).

(Def. 16) Let s be a set. Suppose $s \in$ the carrier of MonSet(L). Then there exists an auxiliary binary relation A_1 on L such that

(i) $A_1 = (\mathrm{Map2Rel}(L))(s)$, and

(ii) for all sets x, y holds $\langle x, y\rangle \in A_1$ iff there exist elements x', y' of L and there exists a map s' from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ such that $x' = x$ and $y' = y$ and $s' = s$ and $x' \in s'(y')$.

Let us consider L. One can check that Map2Rel(L) is monotone.

We now state two propositions:

(26) $\mathrm{Map2Rel}(L) \cdot \mathrm{Rel2Map}(L) = \mathrm{id}_{\mathrm{dom}\,\mathrm{Rel2Map}(L)}$.

(27) $\mathrm{Rel2Map}(L) \cdot \mathrm{Map2Rel}(L) = \mathrm{id}_{\text{the carrier of MonSet}(L)}$.

Let us consider L. Observe that Rel2Map(L) is one-to-one.

The following three propositions are true:

(28) $(\mathrm{Rel2Map}(L))^{-1} = \mathrm{Map2Rel}(L)$.

(29) Rel2Map(L) is isomorphic.

(30) For every complete lattice L and for every element x of L holds $\bigcap\{I, I$ ranges over ideals of L: $x \leqslant \sup I\} = \downarrow x$.

The scheme *LambdaC'* concerns a non empty relational structure $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

> There exists a function f such that $\mathrm{dom}\, f =$ the carrier of $\mathcal{A}$ and for every element x of $\mathcal{A}$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ and if not $\mathcal{P}[x]$, then $f(x) = \mathcal{G}(x)$

for all values of the parameters.

Let L be a semilattice and let I be an ideal of L. The functor DownMap(I) yields a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ and is defined by:

(Def. 17) For every element x of L holds if $x \leqslant \sup I$, then $(\mathrm{DownMap}(I))(x) = {\downarrow}x \cap I$ and if $x \nleqslant \sup I$, then $(\mathrm{DownMap}(I))(x) = {\downarrow}x$.

One can prove the following two propositions:

(31) For every semilattice L and for every ideal I of L holds $\mathrm{DownMap}(I) \in$ the carrier of $\mathrm{MonSet}(L)$.

(32) Let L be an antisymmetric reflexive relational structure with g.l.b.'s, x be an element of L, and D be a non empty lower subset of L. Then $\{x\} \sqcap D = {\downarrow}x \cap D$.

2. Approximating Relations

Let L be a non empty relational structure and let A_1 be a binary relation on L. We say that A_1 is approximating if and only if:

(Def. 18) For every element x of L holds $x = \sup {\downarrow}_{A_1} x$.

Let L be a non empty poset and let m_1 be a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$. We say that m_1 is approximating if and only if:

(Def. 19) For every element x of L there exists a subset i_1 of L such that $i_1 = m_1(x)$ and $x = \sup i_1$.

Next we state two propositions:

(33) For every lower-bounded meet-continuous semilattice L and for every ideal I of L holds $\mathrm{DownMap}(I)$ is approximating.

(34) Every lower-bounded continuous lattice is meet-continuous.

Let us mention that every lower-bounded lattice which is continuous is also meet-continuous.

The following proposition is true

(35) For every lower-bounded continuous lattice L and for every ideal I of L holds $\mathrm{DownMap}(I)$ is approximating.

Let L be a non empty reflexive antisymmetric relational structure. Observe that $\ll_L$ is auxiliary(i).

Let L be a non empty reflexive transitive relational structure. One can check that $\ll_L$ is auxiliary(ii).

Let L be a poset with l.u.b.'s. One can verify that $\ll_L$ is auxiliary(iii).

Let L be an inf-complete non empty poset. Note that $\ll_L$ is auxiliary(iii).

Let L be a lower-bounded antisymmetric reflexive non empty relational structure. Observe that $\ll_L$ is auxiliary(iv).

Next we state two propositions:

(36) For every complete lattice L and for every element x of L holds ${\downarrow}_{\ll_L} x = {\downarrow}x$.

(37) For every lattice L holds $\leqslant_L$ is approximating.

Let L be a lower-bounded continuous lattice. One can verify that $\ll_L$ is approximating.

Let L be a complete lattice. Observe that there exists an auxiliary binary relation on L which is approximating.

Let L be a complete lattice. The functor $\mathrm{App}(L)$ is defined as follows:

(Def. 20) $a \in \mathrm{App}(L)$ iff a is an approximating auxiliary binary relation on L.

Let L be a complete lattice. Note that $\mathrm{App}(L)$ is non empty.

Next we state three propositions:

(38) Let L be a complete lattice and m_1 be a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$. Suppose m_1 is approximating and $m_1 \in$ the carrier of $\mathrm{MonSet}(L)$. Then there exists an approximating auxiliary binary relation A_1 on L such that $A_1 = (\mathrm{Map2Rel}(L))(m_1)$.

(39) For every complete lattice L and for every element x of L holds $\bigcap \{(\mathrm{DownMap}(I))(x) : I$ ranges over ideals of $L\} = \downarrow x$.

(40) Let L be a lower-bounded meet-continuous lattice and x be an element of L. Then $\bigcap \{\downarrow_{A_1} x, A_1$ ranges over auxiliary binary relations on L: $A_1 \in \mathrm{App}(L)\} = \downarrow x$.

In the sequel L denotes a complete lattice.

Next we state two propositions:

(41) L is continuous if and only if for every approximating auxiliary binary relation R on L holds $\ll_L \subseteq R$ and $\ll_L$ is approximating.

(42) L is continuous if and only if the following conditions are satisfied:
  (i) L is meet-continuous, and
  (ii) there exists an approximating auxiliary binary relation R on L such that for every approximating auxiliary binary relation R' on L holds $R \subseteq R'$.

Let L be a non empty relational structure and let A_1 be a binary relation on L. We say that A_1 satisfies strong interpolation property if and only if:

(Def. 21) For all elements x, z of L such that $\langle x, z \rangle \in A_1$ and $x \neq z$ there exists an element y of L such that $\langle x, y \rangle \in A_1$ and $\langle y, z \rangle \in A_1$ and $x \neq y$.

Let L be a non empty relational structure and let A_1 be a binary relation on L. We say that A_1 satisfies interpolation property if and only if:

(Def. 22) For all elements x, z of L such that $\langle x, z \rangle \in A_1$ there exists an element y of L such that $\langle x, y \rangle \in A_1$ and $\langle y, z \rangle \in A_1$.

Next we state two propositions:

(43) Let L be a non empty relational structure, A_1 be a binary relation on L, and x, z be elements of L. If $\langle x, z \rangle \in A_1$ and $x = z$, then there exists an element y of L such that $\langle x, y \rangle \in A_1$ and $\langle y, z \rangle \in A_1$.

(44) Let L be a non empty relational structure and A_1 be a binary relation on L. Suppose A_1 satisfies strong interpolation property. Then A_1 satisfies interpolation property.

Let L be a non empty relational structure. Observe that every binary relation on L which satisfies strong interpolation property satisfies also interpolation property.

In the sequel A_1 is an auxiliary binary relation on L and x, y, z are elements of L.

The following four propositions are true:

(45) Let A_1 be an approximating auxiliary binary relation on L. If $x \nleqslant y$, then there exists an element u of L such that $\langle u, x\rangle \in A_1$ and $u \nleqslant y$.

(46) Let R be an approximating auxiliary binary relation on L. If $\langle x, z\rangle \in R$ and $x \neq z$, then there exists y such that $x \leqslant y$ and $\langle y, z\rangle \in R$ and $x \neq y$.

(47) Let R be an approximating auxiliary binary relation on L. Suppose $x \ll z$ and $x \neq z$. Then there exists an element y of L such that $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$ and $x \neq y$.

(48) For every lower-bounded continuous lattice L holds $\ll_L$ satisfies strong interpolation property.

Let L be a lower-bounded continuous lattice. Observe that $\ll_L$ satisfies strong interpolation property.

Next we state two propositions:

(49) Let L be a lower-bounded continuous lattice and x, y be elements of L. If $x \ll y$, then there exists an element x' of L such that $x \ll x'$ and $x' \ll y$.

(50) Let L be a lower-bounded continuous lattice and x, y be elements of L. Then $x \ll y$ if and only if for every non empty directed subset D of L such that $y \leqslant \sup D$ there exists an element d of L such that $d \in D$ and $x \ll d$.

3. Exercises

Let L be a relational structure, let X be a subset of L, and let R be a binary relation on the carrier of L. We say that X is directed w.r.t. R if and only if:

(Def. 23) For all elements x, y of L such that $x \in X$ and $y \in X$ there exists an element z of L such that $z \in X$ and $\langle x, z\rangle \in R$ and $\langle y, z\rangle \in R$.

We now state the proposition

(51) Let L be a relational structure and X be a subset of L. Suppose X is directed w.r.t. the internal relation of L. Then X is directed.

Let L be a relational structure, let X be a set, let x be an element of L, and let R be a binary relation on the carrier of L. We say that x is maximal w.r.t. X, R if and only if:

(Def. 24) $x \in X$ and it is not true that there exists an element y of L such that $y \in X$ and $y \neq x$ and $\langle x, y\rangle \in R$.

Let L be a relational structure, let X be a set, and let x be an element of L. We say that x is maximal in X if and only if:

(Def. 25) x is maximal w.r.t. X, the internal relation of L.

One can prove the following proposition

(52) Let L be a relational structure, X be a set, and x be an element of L. Then x is maximal in X if and only if the following conditions are satisfied:

(i) $x \in X$, and

(ii) it is not true that there exists an element y of L such that $y \in X$ and $x < y$.

Let L be a relational structure, let X be a set, let x be an element of L, and let R be a binary relation on the carrier of L. We say that x is minimal w.r.t. X, R if and only if:

(Def. 26) $x \in X$ and it is not true that there exists an element y of L such that $y \in X$ and $y \neq x$ and $\langle y, x \rangle \in R$.

Let L be a relational structure, let X be a set, and let x be an element of L. We say that x is minimal in X if and only if:

(Def. 27) x is minimal w.r.t. X, the internal relation of L.

We now state several propositions:

(53) Let L be a relational structure, X be a set, and x be an element of L. Then x is minimal in X if and only if the following conditions are satisfied:

(i) $x \in X$, and

(ii) it is not true that there exists an element y of L such that $y \in X$ and $x > y$.

(54) If A_1 satisfies strong interpolation property, then for every x such that there exists y which is maximal w.r.t. $\downarrow_{A_1} x$, A_1 holds $\langle x, x \rangle \in A_1$.

(55) If for every x such that there exists y which is maximal w.r.t. $\downarrow_{A_1} x$, A_1 holds $\langle x, x \rangle \in A_1$, then A_1 satisfies strong interpolation property.

(56) If A_1 satisfies interpolation property, then for every x holds $\downarrow_{A_1} x$ is directed w.r.t. A_1.

(57) If for every x holds $\downarrow_{A_1} x$ is directed w.r.t. A_1, then A_1 satisfies interpolation property.

(58) Let R be an approximating auxiliary binary relation on L. Suppose R satisfies interpolation property. Then R satisfies strong interpolation property.

Let us consider L. One can verify that every approximating auxiliary binary relation on L which satisfies interpolation property satisfies also strong interpolation property.

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Received October 21, 1996