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# Irreducible and Prime Elements[1]

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**Summary.** In the paper open and order generating subsets are defined. Irreducible and prime elements are also defined. The article includes definitions and facts presented in [16, pp. 68–72].

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The articles [29], [25], [1], [15], [28], [30], [31], [9], [23], [2], [24], [4], [11], [12], [10], [13], [3], [27], [21], [22], [5], [18], [6], [14], [33], [19], [20], [8], [17], [32], [26], and [7] provide the notation and terminology for this paper.

## 1. Preliminaries

In this paper $L$ denotes a lattice and $l$ denotes an element of $L$.

The scheme *NonUniqExD1* concerns a non empty relational structure $\mathcal{A}$, a subset $\mathcal{B}$ of $\mathcal{A}$, a non empty subset $\mathcal{C}$ of $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a function $f$ from $\mathcal{B}$ into $\mathcal{C}$ such that for every element $e$ of $\mathcal{A}$ if $e \in \mathcal{B}$, then there exists an element $u$ of $\mathcal{A}$ such that $u \in \mathcal{C}$ and $u = f(e)$ and $\mathcal{P}[e, u]$

provided the following requirement is met:

- For every element $e$ of $\mathcal{A}$ such that $e \in \mathcal{B}$ there exists an element $u$ of $\mathcal{A}$ such that $u \in \mathcal{C}$ and $\mathcal{P}[e, u]$.

Let $L$ be a lattice, let $A$ be a non empty subset of the carrier of $L$, let $f$ be a function from $A$ into $A$, and let $n$ be an element of $\mathbb{N}$. Then $f^n$ is a function from $A$ into $A$.

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Let $L$ be a lattice, let $C$, $D$ be non empty subsets of the carrier of $L$, let $f$ be a function from $C$ into $D$, and let $c$ be an element of $C$. Then $f(c)$ is an element of $L$.

Let $L$ be a non empty poset. One can check that every chain of $L$ is filtered and directed.

Let us observe that there exists a lattice which is strict, continuous, distributive, and lower-bounded.

Next we state three propositions:

(1) Let $S$, $T$ be semilattices and $f$ be a map from $S$ into $T$. Then $f$ is meet-preserving if and only if for all elements $x$, $y$ of $S$ holds $f(x \sqcap y) = f(x) \sqcap f(y)$.

(2) Let $S$, $T$ be sup-semilattices and $f$ be a map from $S$ into $T$. Then $f$ is join-preserving if and only if for all elements $x$, $y$ of $S$ holds $f(x \sqcup y) = f(x) \sqcup f(y)$.

(3) Let $S$, $T$ be lattices and $f$ be a map from $S$ into $T$. Suppose $T$ is distributive and $f$ is meet-preserving, join-preserving, and one-to-one. Then $S$ is distributive.

Let $S$, $T$ be complete lattices. Observe that there exists a map from $S$ into $T$ which is sups-preserving.

The following proposition is true

(4) Let $S, T$ be complete lattices and $f$ be a sups-preserving map from $S$ into $T$. Suppose $T$ is meet-continuous and $f$ is meet-preserving and one-to-one. Then $S$ is meet-continuous.

## 2. Open sets

Let $L$ be a non empty reflexive relational structure and let $X$ be a subset of $L$. We say that $X$ is open if and only if:

(Def. 1) For every element $x$ of $L$ such that $x \in X$ there exists an element $y$ of $L$ such that $y \in X$ and $y \ll x$.

The following two propositions are true:

(5) Let $L$ be an up-complete lattice and $X$ be an upper subset of $L$. Then $X$ is open if and only if for every element $x$ of $L$ such that $x \in X$ holds $\downarrow\!\!\!\downarrow x \cap X \neq \emptyset$.

(6) Let $L$ be an up-complete lattice and $X$ be an upper subset of $L$. Then $X$ is open if and only if $X = \bigcup\{\uparrow\!\!\!\uparrow x, x$ ranges over elements of $L$: $x \in X\}$.

Let $L$ be an up-complete lower-bounded lattice. Note that there exists a filter of $L$ which is open.

The following three propositions are true:

(7) For every lower-bounded continuous lattice $L$ and for every element $x$ of $L$ holds $\uparrow\!\!\!\uparrow x$ is open.

(8) Let $L$ be a lower-bounded continuous lattice and $x$, $y$ be elements of $L$. If $x \ll y$, then there exists an open filter $F$ of $L$ such that $y \in F$ and $F \subseteq \uparrow x$.

(9) Let $L$ be a complete lattice, $X$ be an open upper subset of $L$, and $x$ be an element of $L$. If $x \in -X$, then there exists an element $m$ of $L$ such that $x \leqslant m$ and $m$ is maximal in $-X$.

## 3. Irreducible elements

Let $G$ be a non empty relational structure and let $g$ be an element of $G$. We say that $g$ is meet-irreducible if and only if:

(Def. 2) For all elements $x$, $y$ of $G$ such that $g = x \sqcap y$ holds $x = g$ or $y = g$.

We introduce $g$ is irreducible as a synonym of $g$ is meet-irreducible.

Let $G$ be a non empty relational structure and let $g$ be an element of $G$. We say that $g$ is join-irreducible if and only if:

(Def. 3) For all elements $x$, $y$ of $G$ such that $g = x \sqcup y$ holds $x = g$ or $y = g$.

Let $L$ be a non empty relational structure. The functor $\mathrm{IRR}(L)$ yielding a subset of $L$ is defined as follows:

(Def. 4) For every element $x$ of $L$ holds $x \in \mathrm{IRR}(L)$ iff $x$ is irreducible.

The following proposition is true

(10) For every upper-bounded antisymmetric non empty relational structure $L$ with g.l.b.'s holds $\top_L$ is irreducible.

Let $L$ be an upper-bounded antisymmetric non empty relational structure with g.l.b.'s. Observe that there exists an element of $L$ which is irreducible.

We now state four propositions:

(11) Let $L$ be a semilattice and $x$ be an element of $L$. Then $x$ is irreducible if and only if for every finite non empty subset $A$ of $L$ such that $x = \inf A$ holds $x \in A$.

(12) For every lattice $L$ and for every element $l$ of $L$ such that $\uparrow l \setminus \{l\}$ is a filter of $L$ holds $l$ is irreducible.

(13) Let $L$ be a lattice, $p$ be an element of $L$, and $F$ be a filter of $L$. If $p$ is maximal in $-F$, then $p$ is irreducible.

(14) Let $L$ be a lower-bounded continuous lattice and $x$, $y$ be elements of $L$. Suppose $y \nleqslant x$. Then there exists an element $p$ of $L$ such that $p$ is irreducible and $x \leqslant p$ and $y \nleqslant p$.

## 4. Order generating sets

Let $L$ be a non empty relational structure and let $X$ be a subset of $L$. We say that $X$ is order-generating if and only if:

(Def. 5) For every element $x$ of $L$ holds $\inf \uparrow x \cap X$ exists in $L$ and $x = \inf(\uparrow x \cap X)$.

The following propositions are true:

(15) Let $L$ be an up-complete lower-bounded lattice and $X$ be a subset of $L$. Then $X$ is order-generating if and only if for every element $l$ of $L$ there exists a subset $Y$ of $X$ such that $l = \bigsqcap_L Y$.

(16) Let $L$ be an up-complete lower-bounded lattice and $X$ be a subset of $L$. Then $X$ is order-generating if and only if for every subset $Y$ of $L$ such that $X \subseteq Y$ and for every subset $Z$ of $Y$ holds $\bigsqcap_L Z \in Y$ holds the carrier of $L = Y$.

(17) Let $L$ be an up-complete lower-bounded lattice and $X$ be a subset of $L$. Then $X$ is order-generating if and only if for all elements $l_1$, $l_2$ of $L$ such that $l_2 \not\leqslant l_1$ there exists an element $p$ of $L$ such that $p \in X$ and $l_1 \leqslant p$ and $l_2 \not\leqslant p$.

(18) Let $L$ be a lower-bounded continuous lattice and $X$ be a subset of $L$. If $X = \mathrm{IRR}(L) \setminus \{\top_L\}$, then $X$ is order-generating.

(19) Let $L$ be a lower-bounded continuous lattice and $X$, $Y$ be subsets of $L$. If $X$ is order-generating and $X \subseteq Y$, then $Y$ is order-generating.

## 5. Prime elements

Let $L$ be a non empty relational structure and let $l$ be an element of $L$. We say that $l$ is prime if and only if:

(Def. 6) For all elements $x$, $y$ of $L$ such that $x \sqcap y \leqslant l$ holds $x \leqslant l$ or $y \leqslant l$.

Let $L$ be a non empty relational structure. The functor $\mathrm{PRIME}(L)$ yielding a subset of $L$ is defined by:

(Def. 7) For every element $x$ of $L$ holds $x \in \mathrm{PRIME}(L)$ iff $x$ is prime.

Let $L$ be a non empty relational structure and let $l$ be an element of $L$. We say that $l$ is co-prime if and only if:

(Def. 8) $l^{\smile}$ is prime.

We now state two propositions:

(20) For every upper-bounded antisymmetric non empty relational structure $L$ holds $\top_L$ is prime.

(21) For every lower-bounded antisymmetric non empty relational structure $L$ holds $\bot_L$ is co-prime.

Let $L$ be an upper-bounded antisymmetric non empty relational structure. Note that there exists an element of $L$ which is prime.

The following propositions are true:

(22) Let $L$ be a semilattice and $l$ be an element of $L$. Then $l$ is prime if and only if for every finite non empty subset $A$ of $L$ such that $l \geqslant \inf A$ there exists an element $a$ of $L$ such that $a \in A$ and $l \geqslant a$.

(23) Let $L$ be a sup-semilattice and $x$ be an element of $L$. Then $x$ is co-prime if and only if for every finite non empty subset $A$ of $L$ such that $x \leqslant \sup A$ there exists an element $a$ of $L$ such that $a \in A$ and $x \leqslant a$.

(24) For every lattice $L$ and for every element $l$ of $L$ such that $l$ is prime holds $l$ is irreducible.

(25) Let given $l$. Then $l$ is prime if and only if for arbitrary $x$ and for every map $f$ from $L$ into $2_{\subseteq}^{\{x\}}$ such that for every element $p$ of $L$ holds $f(p) = \emptyset$ iff $p \leqslant l$ holds $f$ is meet-preserving and join-preserving.

(26) Let $L$ be an upper-bounded lattice and $l$ be an element of $L$. If $l \neq \top_L$, then $l$ is prime iff $-{\downarrow}l$ is a filter of $L$.

(27) For every distributive lattice $L$ and for every element $l$ of $L$ holds $l$ is prime iff $l$ is irreducible.

(28) For every distributive lattice $L$ holds $\mathrm{PRIME}(L) = \mathrm{IRR}(L)$.

(29) Let $L$ be a Boolean lattice and $l$ be an element of $L$. Suppose $l \neq \top_L$. Then $l$ is prime if and only if for every element $x$ of $L$ such that $x > l$ holds $x = \top_L$.

(30) Let $L$ be a continuous distributive lower-bounded lattice and $l$ be an element of $L$. Suppose $l \neq \top_L$. Then $l$ is prime if and only if there exists an open filter $F$ of $L$ such that $l$ is maximal in $-F$.

(31) Let $L$ be a relational structure and $X$ be a subset of the carrier of $L$. Then $\chi_{X,\text{the carrier of } L}$ is a map from $L$ into $2_{\subseteq}^{\{\emptyset\}}$.

(32) Let $L$ be a non empty relational structure and $p$, $x$ be elements of $L$. Then $\chi_{-{\downarrow}p,\text{the carrier of } L}(x) = \emptyset$ if and only if $x \leqslant p$.

(33) Let $L$ be an upper-bounded lattice, $f$ be a map from $L$ into $2_{\subseteq}^{\{\emptyset\}}$, and $p$ be a prime element of $L$. Suppose $\chi_{-{\downarrow}p,\text{the carrier of } L} = f$. Then $f$ is meet-preserving and join-preserving.

(34) For every complete lattice $L$ such that $\mathrm{PRIME}(L)$ is order-generating holds $L$ is distributive and meet-continuous.

(35) For every lower-bounded continuous lattice $L$ holds $L$ is distributive iff $\mathrm{PRIME}(L)$ is order-generating.

(36) For every lower-bounded continuous lattice $L$ holds $L$ is distributive iff $L$ is Heyting.

(37) Let $L$ be a continuous complete lattice. Suppose that for every element $l$ of $L$ there exists a subset $X$ of $L$ such that $l = \sup X$ and for every element $x$ of $L$ such that $x \in X$ holds $x$ is co-prime. Let $l$ be an element of $L$. Then $l = \bigsqcup_L({\downarrow}l \cap \mathrm{PRIME}(L^{\mathrm{op}}))$.

(38) Let $L$ be a complete lattice. Then $L$ is completely-distributive if and only if the following conditions are satisfied:
  (i) $L$ is continuous, and
  (ii) for every element $l$ of $L$ there exists a subset $X$ of $L$ such that $l = \sup X$ and for every element $x$ of $L$ such that $x \in X$ holds $x$ is co-prime.

(39) Let $L$ be a complete lattice. Then $L$ is completely-distributive if and only if $L$ is distributive and continuous and $L^{\mathrm{op}}$ is continuous.

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