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Prime Ideals and Filters[1]

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Summary. The part of [12, pp. 73–77], i.e. definitions and propositions 3.16–3.27, is formalized in the paper.

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The notation and terminology used in this paper are introduced in the following articles: [22], [25], [8], [24], [19], [26], [27], [7], [11], [6], [20], [10], [15], [21], [23], [1], [2], [3], [14], [9], [16], [17], [5], [4], [18], and [13].

1. The lattice of subsets

One can prove the following propositions:

(1) For every complete lattice L and for every ideal I of L holds $\perp_L \in I$.

(2) For every upper-bounded non empty poset L and for every filter F of L holds $\top_L \in F$.

(3) For every complete lattice L and for all sets X, Y such that $X \subseteq Y$ holds $\bigsqcup_L X \leqslant \bigsqcup_L Y$ and $\bigsqcap_L X \geqslant \bigsqcap_L Y$.

(4) For every set X holds the carrier of $2^X_\subseteq = 2^X$.

(5) For every bounded antisymmetric non empty relational structure L holds L is trivial iff $\top_L = \perp_L$.

Let X be a set. Note that $2^X_\subseteq$ is Boolean.

Let X be a non empty set. Note that $2^X_\subseteq$ is non trivial.

We now state three propositions:

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(6) For every upper-bounded non empty poset L holds $\{\top_L\} = \uparrow(\top_L)$.

(7) For every lower-bounded non empty poset L holds $\{\bot_L\} = \downarrow(\bot_L)$.

(8) For every lower-bounded non empty poset L and for every filter F of L holds F is proper iff $\bot_L \notin F$.

One can verify that there exists a lattice which is non trivial, Boolean, and strict.

Let L be a non trivial upper-bounded non empty poset. One can check that there exists a filter of L which is proper.

Next we state several propositions:

(9) For every set X and for every element a of $2^X_\subseteq$ holds $\neg a = X \setminus a$.

(10) Let X be a set and Y be a subset of $2^X_\subseteq$. Then Y is lower if and only if for all sets x, y such that $x \subseteq y$ and $y \in Y$ holds $x \in Y$.

(11) Let X be a set and Y be a subset of $2^X_\subseteq$. Then Y is upper if and only if for all sets x, y such that $x \subseteq y$ and $y \subseteq X$ and $x \in Y$ holds $y \in Y$.

(12) Let X be a set and Y be a lower subset of $2^X_\subseteq$. Then Y is directed if and only if for all sets x, y such that $x \in Y$ and $y \in Y$ holds $x \cup y \in Y$.

(13) Let X be a set and Y be an upper subset of $2^X_\subseteq$. Then Y is filtered if and only if for all sets x, y such that $x \in Y$ and $y \in Y$ holds $x \cap y \in Y$.

(14) Let X be a set and Y be a non empty lower subset of $2^X_\subseteq$. Then Y is directed if and only if for every finite family Z of subsets of X such that $Z \subseteq Y$ holds $\bigcup Z \in Y$.

(15) Let X be a set and Y be a non empty upper subset of $2^X_\subseteq$. Then Y is filtered if and only if for every finite family Z of subsets of X such that $Z \subseteq Y$ holds $\mathrm{Intersect}(Z) \in Y$.

2. Prime ideals and filters

Let L be a poset with g.l.b.'s and let I be an ideal of L. We say that I is prime if and only if:

(Def. 1) For all elements x, y of L such that $x \sqcap y \in I$ holds $x \in I$ or $y \in I$.

One can prove the following proposition

(16) Let L be a poset with g.l.b.'s and I be an ideal of L. Then I is prime if and only if for every finite non empty subset A of L such that $\inf A \in I$ there exists an element a of L such that $a \in A$ and $a \in I$.

Let L be a lattice. Note that there exists an ideal of L which is prime.

Next we state the proposition

(17) Let L_1, L_2 be lattices. Suppose the relational structure of L_1 = the relational structure of L_2. Let x be a set. If x is a prime ideal of L_1, then x is a prime ideal of L_2.

Let L be a poset with l.u.b.'s and let F be a filter of L. We say that F is prime if and only if:

(Def. 2) For all elements x, y of L such that $x \sqcup y \in F$ holds $x \in F$ or $y \in F$.

Next we state the proposition

(18) Let L be a poset with l.u.b.'s and F be a filter of L. Then F is prime if and only if for every finite non empty subset A of L such that $\sup A \in F$ there exists an element a of L such that $a \in A$ and $a \in F$.

Let L be a lattice. One can verify that there exists a filter of L which is prime.

The following propositions are true:

(19) Let L_1, L_2 be lattices. Suppose the relational structure of L_1 = the relational structure of L_2. Let x be a set. If x is a prime filter of L_1, then x is a prime filter of L_2.

(20) Let L be a lattice and x be a set. Then x is a prime ideal of L if and only if x is a prime filter of L^{op}.

(21) Let L be a lattice and x be a set. Then x is a prime filter of L if and only if x is a prime ideal of L^{op}.

(22) Let L be a poset with g.l.b.'s and I be an ideal of L. Then I is prime if and only if one of the following conditions is satisfied:
  (i) $-I$ is a filter of L, or
  (ii) $-I = \emptyset$.

(23) For every lattice L and for every ideal I of L holds I is prime iff $I \in \mathrm{PRIME}(\langle \mathrm{Ids}(L), \subseteq \rangle)$.

(24) Let L be a Boolean lattice and F be a filter of L. Then F is prime if and only if for every element a of L holds $a \in F$ or $\neg a \in F$.

(25) Let X be a set and F be a filter of $2^X_{\subseteq}$. Then F is prime if and only if for every subset A of X holds $A \in F$ or $X \setminus A \in F$.

Let L be a non empty poset and let F be a filter of L. We say that F is ultra if and only if:

(Def. 3) F is proper and for every filter G of L such that $F \subseteq G$ holds $F = G$ or G = the carrier of L.

Let L be a non empty poset. Note that every filter of L which is ultra is also proper.

The following propositions are true:

(26) For every Boolean lattice L and for every filter F of L holds F is proper and prime iff F is ultra.

(27) Let L be a distributive lattice, I be an ideal of L, and F be a filter of L. Suppose I misses F. Then there exists an ideal P of L such that P is prime and $I \subseteq P$ and P misses F.

(28) Let L be a distributive lattice, I be an ideal of L, and x be an element of L. If $x \notin I$, then there exists an ideal P of L such that P is prime and $I \subseteq P$ and $x \notin P$.

(29) Let L be a distributive lattice, I be an ideal of L, and F be a filter of L. Suppose I misses F. Then there exists a filter P of L such that P is

prime and $F \subseteq P$ and I misses P.

(30) Let L be a non trivial Boolean lattice and F be a proper filter of L. Then there exists a filter G of L such that $F \subseteq G$ and G is ultra.

3. Cluster points of a filter of sets

Let T be a topological space and let F, x be sets. We say that x is a cluster point of F, T if and only if:

(Def. 4) For every subset A of T such that A is open and $x \in A$ and for every set B such that $B \in F$ holds A meets B.

We say that x is a convergence point of F, T if and only if:

(Def. 5) For every subset A of T such that A is open and $x \in A$ holds $A \in F$.

Let X be a non empty set. Note that there exists a filter of $2_{\subseteq}^{X}$ which is ultra.

We now state several propositions:

(31) Let T be a non empty topological space, F be an ultra filter of $2_{\subseteq}^{\text{the carrier of } T}$, and p be a set. Then p is a cluster point of F, T if and only if p is a convergence point of F, T.

(32) Let T be a non empty topological space and x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose $x \ll y$. Let F be a proper filter of $2_{\subseteq}^{\text{the carrier of } T}$. Suppose $x \in F$. Then there exists an element p of T such that $p \in y$ and p is a cluster point of F, T.

(33) Let T be a non empty topological space and x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose $x \ll y$. Let F be an ultra filter of $2_{\subseteq}^{\text{the carrier of } T}$. Suppose $x \in F$. Then there exists an element p of T such that $p \in y$ and p is a convergence point of F, T.

(34) Let T be a non empty topological space and x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose that

(i) $x \subseteq y$, and

(ii) for every ultra filter F of $2_{\subseteq}^{\text{the carrier of } T}$ such that $x \in F$ there exists an element p of T such that $p \in y$ and p is a convergence point of F, T.

Then $x \ll y$.

(35) Let T be a non empty topological space, B be a prebasis of T, and x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose $x \subseteq y$. Then $x \ll y$ if and only if for every subset F of B such that $y \subseteq \bigcup F$ there exists a finite subset G of F such that $x \subseteq \bigcup G$.

(36) Let L be a distributive complete lattice and x, y be elements of L. Then $x \ll y$ if and only if for every prime ideal P of L such that $y \leqslant \sup P$ holds $x \in P$.

(37) For every lattice L and for every element p of L such that p is prime holds $\downarrow p$ is prime.

4. Pseudo prime elements

Let L be a lattice and let p be an element of L. We say that p is pseudoprime if and only if:

(Def. 6) There exists a prime ideal P of L such that $p = \sup P$.

We now state several propositions:

(38) For every lattice L and for every element p of L such that p is prime holds p is pseudoprime.

(39) Let L be a continuous lattice and p be an element of L. Suppose p is pseudoprime. Let A be a finite non empty subset of L. If $\inf A \ll p$, then there exists an element a of L such that $a \in A$ and $a \leqslant p$.

(40) Let L be a continuous lattice and p be an element of L. Suppose that
(i) $p \neq \top_L$ or $\top_L$ is not compact, and
(ii) for every finite non empty subset A of L such that $\inf A \ll p$ there exists an element a of L such that $a \in A$ and $a \leqslant p$.
Then $\uparrow\mathrm{fininfs}(-\downarrow p)$ misses $⇣p$.

(41) Let L be a continuous lattice. Suppose $\top_L$ is compact. Then
(i) for every finite non empty subset A of L such that $\inf A \ll \top_L$ there exists an element a of L such that $a \in A$ and $a \leqslant \top_L$, and
(ii) $\uparrow\mathrm{fininfs}(-\downarrow(\top_L))$ meets $⇣(\top_L)$.

(42) Let L be a continuous lattice and p be an element of L. Suppose $\uparrow\mathrm{fininfs}(-\downarrow p)$ misses $⇣p$. Let A be a finite non empty subset of L. If $\inf A \ll p$, then there exists an element a of L such that $a \in A$ and $a \leqslant p$.

(43) Let L be a distributive continuous lattice and p be an element of L. If $\uparrow\mathrm{fininfs}(-\downarrow p)$ misses $⇣p$, then p is pseudoprime.

Let L be a non empty relational structure and let R be a binary relation on the carrier of L. We say that R is multiplicative if and only if:

(Def. 7) For all elements a, x, y of L such that $\langle a, x\rangle \in R$ and $\langle a, y\rangle \in R$ holds $\langle a, x \sqcap y\rangle \in R$.

Let L be a lower-bounded sup-semilattice, let R be an auxiliary binary relation on L, and let x be an element of L. Observe that $⇡_R x$ is upper.

We now state several propositions:

(44) Let L be a lower-bounded lattice and R be an auxiliary binary relation on L. Then R is multiplicative if and only if for every element x of L holds $⇡_R x$ is filtered.

(45) Let L be a lower-bounded lattice and R be an auxiliary binary relation on L. Then R is multiplicative if and only if for all elements a, b, x, y of L such that $\langle a, x\rangle \in R$ and $\langle b, y\rangle \in R$ holds $\langle a \sqcap b, x \sqcap y\rangle \in R$.

(46) Let L be a lower-bounded lattice and R be an auxiliary binary relation on L. Then R is multiplicative if and only if for every full relational

substructure S of $[: L, L :]$ such that the carrier of $S = R$ holds S is meet-inheriting.

(47) Let L be a lower-bounded lattice and R be an auxiliary binary relation on L. Then R is multiplicative if and only if $\downarrow R$ is meet-preserving.

(48) Let L be a continuous lower-bounded lattice. Suppose $\ll_L$ is multiplicative. Let p be an element of L. Then p is pseudoprime if and only if for all elements a, b of L such that $a \sqcap b \ll p$ holds $a \leqslant p$ or $b \leqslant p$.

(49) Let L be a continuous lower-bounded lattice. Suppose $\ll_L$ is multiplicative. Let p be an element of L. If p is pseudoprime, then p is prime.

(50) Let L be a distributive continuous lower-bounded lattice. Suppose that for every element p of L such that p is pseudoprime holds p is prime. Then $\ll_L$ is multiplicative.

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