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On the Topological Properties of Meet-Continuous Lattices[1]

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Summary. This work is continuation of formalization of [12]. Proposition 4.4 from Chapter 0 is proved.

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The terminology and notation used in this paper are introduced in the following papers: [30], [24], [25], [27], [31], [32], [33], [6], [11], [21], [15], [7], [23], [34], [10], [29], [9], [5], [4], [28], [22], [1], [13], [16], [2], [3], [14], [35], [8], [17], [20], [18], [19], and [26].

1. Preliminaries

Let L be a non empty relational structure. One can check that id_L is monotone.

Let S, T be non empty relational structures and let f be a map from S into T. Let us observe that f is antitone if and only if:

(Def. 1) For all elements x, y of S such that $x \leqslant y$ holds $f(x) \geqslant f(y)$.

Next we state several propositions:

(1) Let S, T be relational structures, K, L be non empty relational structures, f be a map from S into T, and g be a map from K into L. Suppose that

(i) the relational structure of $S =$ the relational structure of K,

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(ii) the relational structure of T = the relational structure of L,
(iii) $f = g$, and
(iv) f is monotone.

Then g is monotone.

(2) Let S, T be relational structures, K, L be non empty relational structures, f be a map from S into T, and g be a map from K into L. Suppose that

(i) the relational structure of S = the relational structure of K,
(ii) the relational structure of T = the relational structure of L,
(iii) $f = g$, and
(iv) f is antitone.

Then g is antitone.

(3) Let A, B be 1-sorted structures, F be a family of subsets of A, and G be a family of subsets of B. Suppose the carrier of A = the carrier of B and $F = G$ and F is a cover of A. Then G is a cover of B.

(4) For every antisymmetric reflexive relational structure L with l.u.b.'s and for every element x of L holds $\uparrow x = \{x\} \sqcup \Omega_L$.

(5) For every antisymmetric reflexive relational structure L with g.l.b.'s and for every element x of L holds $\downarrow x = \{x\} \sqcap \Omega_L$.

(6) For every antisymmetric reflexive relational structure L with g.l.b.'s and for every element y of L holds $(y \sqcap \square)^{\circ} \uparrow y = \{y\}$.

(7) For every antisymmetric reflexive relational structure L with g.l.b.'s and for every element x of L holds $(x \sqcap \square)^{-1}(\{x\}) = \uparrow x$.

(8) For every non empty 1-sorted structure T holds every non empty net structure N over T is eventually in rng (the mapping of N).

Let L be a non empty reflexive relational structure, let D be a non empty directed subset of L, and let n be a function from D into the carrier of L. One can verify that $\langle D, (\text{the internal relation of } L) |^2 D, n\rangle$ is directed.

Let L be a non empty reflexive transitive relational structure, let D be a non empty directed subset of L, and let n be a function from D into the carrier of L. One can check that $\langle D, (\text{the internal relation of } L) |^2 D, n\rangle$ is transitive.

The following propositions are true:

(9) For every non empty reflexive transitive relational structure L such that for every element x of L and for every net N in L such that N is eventually-directed holds $x \sqcap \sup N = \sup\{x\} \sqcap \operatorname{rng} \operatorname{netmap}(N, L)$ holds L satisfies MC.

(10) Let L be a non empty relational structure, a be an element of L, and N be a net in L. Then $a \sqcap N$ is a net in L.

Let L be a non empty relational structure, let x be an element of L, and let N be a net in L. Then $x \sqcap N$ is a strict net in L.

Let L be a non empty relational structure, let x be an element of L, and let N be a non empty reflexive net structure over L. Observe that $x \sqcap N$ is reflexive.

Let L be a non empty relational structure, let x be an element of L, and let N be a non empty antisymmetric net structure over L. Note that $x \sqcap N$ is antisymmetric.

Let L be a non empty relational structure, let x be an element of L, and let N be a non empty transitive net structure over L. Note that $x \sqcap N$ is transitive.

Let L be a non empty relational structure, let J be a set, and let f be a function from J into the carrier of L. Observe that $\mathrm{FinSups}(f)$ is transitive.

2. The Operations Defined on Nets

Let L be a non empty relational structure and let N be a net structure over L. The functor $\inf N$ yielding an element of L is defined as follows:

(Def. 2) $\inf N = \mathrm{Inf}(\text{the mapping of } N)$.

Let L be a relational structure and let N be a net structure over L. We say that $\sup N$ exists if and only if:

(Def. 3) $\mathrm{Sup}\,\mathrm{rng}\,(\text{the mapping of } N)$ exists in L.

We say that $\inf N$ exists if and only if:

(Def. 4) $\mathrm{Inf}\,\mathrm{rng}\,(\text{the mapping of } N)$ exists in L.

Let L be a relational structure. The functor $\langle L; \mathrm{id}\rangle$ yields a strict net structure over L and is defined by:

(Def. 5) The relational structure of $\langle L; \mathrm{id}\rangle$ = the relational structure of L and the mapping of $\langle L; \mathrm{id}\rangle = \mathrm{id}_L$.

Let L be a non empty relational structure. Observe that $\langle L; \mathrm{id}\rangle$ is non empty.

Let L be a reflexive relational structure. One can check that $\langle L; \mathrm{id}\rangle$ is reflexive.

Let L be an antisymmetric relational structure. Note that $\langle L; \mathrm{id}\rangle$ is antisymmetric.

Let L be a transitive relational structure. Observe that $\langle L; \mathrm{id}\rangle$ is transitive.

Let L be a relational structure with l.u.b.'s. One can verify that $\langle L; \mathrm{id}\rangle$ is directed.

Let L be a directed relational structure. Note that $\langle L; \mathrm{id}\rangle$ is directed.

Let L be a non empty relational structure. One can verify that $\langle L; \mathrm{id}\rangle$ is monotone and eventually-directed.

Let L be a relational structure. The functor $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$ yields a strict net structure over L and is defined by the conditions (Def. 6).

(Def. 6)(i) The carrier of $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$ = the carrier of L,

(ii) the internal relation of $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$ = (the internal relation of $L)^{\smile}$, and

(iii) the mapping of $\langle L^{\mathrm{op}}; \mathrm{id}\rangle = \mathrm{id}_L$.

Next we state the proposition

(11) For every relational structure L holds the relational structure of $L^{\smile}$ = the relational structure of $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$.

Let L be a non empty relational structure. One can check that $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$ is non empty.

Let L be a reflexive relational structure. Observe that $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$ is reflexive.

Let L be an antisymmetric relational structure. Observe that $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$ is antisymmetric.

Let L be a transitive relational structure. Note that $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$ is transitive.

Let L be a relational structure with g.l.b.'s. Note that $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$ is directed.

Let L be a non empty relational structure. Note that $\langle L^{\mathrm{op}}; \mathrm{id}\rangle$ is antitone and eventually-filtered.

Let L be a non empty 1-sorted structure, let N be a non empty net structure over L, and let i be an element of N. The functor $N\restriction i$ yields a strict net structure over L and is defined by the conditions (Def. 7).

(Def. 7)(i) For every set x holds $x \in$ the carrier of $N\restriction i$ iff there exists an element y of N such that $y = x$ and $i \leqslant y$,

(ii) the internal relation of $N\restriction i =$ (the internal relation of N) $|^2$ (the carrier of $N\restriction i$), and

(iii) the mapping of $N\restriction i =$ (the mapping of N)$\restriction$(the carrier of $N\restriction i$).

We now state three propositions:

(12) Let L be a non empty 1-sorted structure, N be a non empty net structure over L, and i be an element of N. Then the carrier of $N\restriction i = \{y, y$ ranges over elements of N: $i \leqslant y\}$.

(13) Let L be a non empty 1-sorted structure, N be a non empty net structure over L, and i be an element of N. Then the carrier of $N\restriction i \subseteq$ the carrier of N.

(14) Let L be a non empty 1-sorted structure, N be a non empty net structure over L, and i be an element of N. Then $N\restriction i$ is a full structure of a subnet of N.

Let L be a non empty 1-sorted structure, let N be a non empty reflexive net structure over L, and let i be an element of N. Note that $N\restriction i$ is non empty and reflexive.

Let L be a non empty 1-sorted structure, let N be a non empty directed net structure over L, and let i be an element of N. Note that $N\restriction i$ is non empty.

Let L be a non empty 1-sorted structure, let N be a non empty reflexive antisymmetric net structure over L, and let i be an element of N. Observe that $N\restriction i$ is antisymmetric.

Let L be a non empty 1-sorted structure, let N be a non empty directed antisymmetric net structure over L, and let i be an element of N. Note that $N\restriction i$ is antisymmetric.

Let L be a non empty 1-sorted structure, let N be a non empty reflexive transitive net structure over L, and let i be an element of N. One can verify that $N\restriction i$ is transitive.

Let L be a non empty 1-sorted structure, let N be a net in L, and let i be an element of N. Note that $N\restriction i$ is transitive and directed.

Next we state three propositions:

(15) Let L be a non empty 1-sorted structure, N be a non empty reflexive net structure over L, i, x be elements of N, and x_1 be an element of $N{\restriction}i$. If $x = x_1$, then $N(x) = (N{\restriction}i)(x_1)$.

(16) Let L be a non empty 1-sorted structure, N be a non empty directed net structure over L, i, x be elements of N, and x_1 be an element of $N{\restriction}i$. If $x = x_1$, then $N(x) = (N{\restriction}i)(x_1)$.

(17) Let L be a non empty 1-sorted structure, N be a net in L, and i be an element of N. Then $N{\restriction}i$ is a subnet of N.

Let T be a non empty 1-sorted structure and let N be a net in T. Observe that there exists a subnet of N which is strict.

Let L be a non empty 1-sorted structure, let N be a net in L, and let i be an element of N. Then $N{\restriction}i$ is a strict subnet of N.

Let S be a non empty 1-sorted structure, let T be a 1-sorted structure, let f be a map from S into T, and let N be a net structure over S. The functor $f \cdot N$ yielding a strict net structure over T is defined by the conditions (Def. 8).

(Def. 8)(i) The relational structure of $f \cdot N =$ the relational structure of N, and

(ii) the mapping of $f \cdot N = f \cdot$ the mapping of N.

Let S be a non empty 1-sorted structure, let T be a 1-sorted structure, let f be a map from S into T, and let N be a non empty net structure over S. One can verify that $f \cdot N$ is non empty.

Let S be a non empty 1-sorted structure, let T be a 1-sorted structure, let f be a map from S into T, and let N be a reflexive net structure over S. Observe that $f \cdot N$ is reflexive.

Let S be a non empty 1-sorted structure, let T be a 1-sorted structure, let f be a map from S into T, and let N be an antisymmetric net structure over S. Observe that $f \cdot N$ is antisymmetric.

Let S be a non empty 1-sorted structure, let T be a 1-sorted structure, let f be a map from S into T, and let N be a transitive net structure over S. Note that $f \cdot N$ is transitive.

Let S be a non empty 1-sorted structure, let T be a 1-sorted structure, let f be a map from S into T, and let N be a directed net structure over S. Note that $f \cdot N$ is directed.

One can prove the following proposition

(18) Let L be a non empty relational structure, N be a non empty net structure over L, and x be an element of L. Then $(x \sqcap \Box) \cdot N = x \sqcap N$.

3. The Properties of Topological Spaces

The following two propositions are true:

(19) Let S, T be topological structures, F be a family of subsets of S, and G be a family of subsets of T. Suppose the topological structure of $S =$ the topological structure of T and $F = G$ and F is open. Then G is open.

(20) Let S, T be topological structures, F be a family of subsets of S, and G be a family of subsets of T. Suppose the topological structure of $S =$ the topological structure of T and $F = G$ and F is closed. Then G is closed.

Let a be a set. Note that $\{a\}_{\rm top}$ is discrete.

We consider FR-structures as extensions of topological structure and relational structure as systems

⟨ a carrier, a internal relation, a topology ⟩,

where the carrier is a set, the internal relation is a binary relation on the carrier, and the topology is a family of subsets of the carrier.

Let A be a non empty set, let R be a relation between A and A, and let T be a family of subsets of A. Note that $\langle A, R, T\rangle$ is non empty.

Let x be a set, let R be a binary relation on $\{x\}$, and let T be a family of subsets of $\{x\}$. Note that $\langle \{x\}, R, T\rangle$ is trivial.

Let X be a set, let O be an order in X, and let T be a family of subsets of X. Observe that $\langle X, O, T\rangle$ is reflexive transitive and antisymmetric.

Let us observe that there exists a FR-structure which is trivial, reflexive, non empty, discrete, strict, and finite.

A TopLattice is a reflexive transitive antisymmetric topological space-like FR-structure with g.l.b.'s and l.u.b.'s.

Let us observe that there exists a non empty TopLattice which is strict, trivial, discrete, finite, compact, and Hausdorff.

Let T be a Hausdorff non empty topological space. One can check that every non empty subspace of T is Hausdorff.

One can prove the following propositions:

(21) For every non empty topological space T and for every point p of T holds every element of the open neighbourhoods of p is a neighbourhood of p.

(22) Let T be a non empty topological space, p be a point of T, and A, B be elements of the open neighbourhoods of p. Then $A \cap B$ is an element of the open neighbourhoods of p.

(23) Let T be a non empty topological space, p be a point of T, and A, B be elements of the open neighbourhoods of p. Then $A \cup B$ is an element of the open neighbourhoods of p.

(24) Let T be a non empty topological space, p be an element of the carrier of T, and N be a net in T. Suppose $p \in \operatorname{Lim} N$. Let S be a subset of the carrier of T. If $S = \operatorname{rng}$ (the mapping of N), then $p \in \overline{S}$.

(25) Let T be a Hausdorff non empty TopLattice, N be a convergent net in T, and f be a map from T into T. If f is continuous, then $f(\lim N) \in \operatorname{Lim}(f \cdot N)$.

(26) Let T be a Hausdorff non empty TopLattice, N be a convergent net in T, and x be an element of T. If $x \sqcap \square$ is continuous, then $x \sqcap \lim N \in \operatorname{Lim}(x \sqcap N)$.

(27) Let S be a Hausdorff non empty TopLattice and x be an element of S. If for every element a of S holds $a \sqcap \square$ is continuous, then $\uparrow x$ is closed.

(28) Let S be a compact Hausdorff non empty TopLattice and x be an element of S. If for every element b of S holds $b \sqcap \square$ is continuous, then $\downarrow x$ is closed.

4. The Cluster Points of Nets

Let T be a TopLattice, let N be a non empty net structure over T, and let p be a point of T. We say that p is a cluster point of N if and only if:

(Def. 9) For every neighbourhood O of p holds N is often in O.

Next we state several propositions:

(29) Let L be a non empty TopLattice, N be a net in L, and c be a point of L. If $c \in \operatorname{Lim} N$, then c is a cluster point of N.

(30) Let T be a compact Hausdorff non empty TopLattice and N be a net in T. Then there exists a point c of T such that c is a cluster point of N.

(31) Let L be a non empty TopLattice, N be a net in L, M be a subnet of N, and c be a point of L. If c is a cluster point of M, then c is a cluster point of N.

(32) Let T be a non empty TopLattice, N be a net in T, and x be a point of T. Suppose x is a cluster point of N. Then there exists a subnet M of N such that $x \in \operatorname{Lim} M$.

(33) Let L be a compact Hausdorff non empty TopLattice and N be a net in L. Suppose that for all points c, d of L such that c is a cluster point of N and d is a cluster point of N holds $c = d$. Let s be a point of L. If s is a cluster point of N, then $s \in \operatorname{Lim} N$.

(34) Let S be a non empty TopLattice, c be a point of S, N be a net in S, and A be a subset of S. Suppose c is a cluster point of N and A is closed and $\operatorname{rng}(\text{the mapping of } N) \subseteq A$. Then $c \in A$.

(35) Let S be a compact Hausdorff non empty TopLattice, c be a point of S, and N be a net in S. Suppose for every element x of S holds $x \sqcap \square$ is continuous and N is eventually-directed and c is a cluster point of N. Then $c = \sup N$.

(36) Let S be a compact Hausdorff non empty TopLattice, c be a point of S, and N be a net in S. Suppose for every element x of S holds $x \sqcap \square$ is continuous and N is eventually-filtered and c is a cluster point of N. Then $c = \inf N$.

5. On The Topological Properties of Meet-Continuous Lattices

Next we state several propositions:

(37) Let S be a Hausdorff non empty TopLattice. Suppose that

(i) for every net N in S such that N is eventually-directed holds sup N exists and $\sup N \in \mathrm{Lim}\, N$, and

(ii) for every element x of S holds $x \sqcap \Box$ is continuous.

Then S is meet-continuous.

(38) Let S be a compact Hausdorff non empty TopLattice. Suppose that for every element x of S holds $x \sqcap \Box$ is continuous. Let N be a net in S. If N is eventually-directed, then sup N exists and $\sup N \in \mathrm{Lim}\, N$.

(39) Let S be a compact Hausdorff non empty TopLattice. Suppose that for every element x of S holds $x \sqcap \Box$ is continuous. Let N be a net in S. If N is eventually-filtered, then inf N exists and $\inf N \in \mathrm{Lim}\, N$.

(40) Let S be a compact Hausdorff non empty TopLattice. If for every element x of S holds $x \sqcap \Box$ is continuous, then S is bounded.

(41) Let S be a compact Hausdorff non empty TopLattice. Suppose that for every element x of S holds $x \sqcap \Box$ is continuous. Then S is meet-continuous.

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