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Baire Spaces, Sober Spaces[1]

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Summary. In the article concepts and facts necessary to continue formalization of theory of continuous lattices according to [10] are introduced.

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The notation and terminology used here are introduced in the following papers: [17], [22], [21], [23], [7], [13], [2], [1], [3], [5], [9], [19], [16], [14], [24], [11], [12], [15], [6], [18], [20], [8], and [4].

1. Preliminaries

One can prove the following propositions:

(1) For all sets X, A, B such that $A \in \mathrm{Fin}\, X$ and $B \subseteq A$ holds $B \in \mathrm{Fin}\, X$.

(2) For every set X and for every family F of subsets of X such that $F \subseteq \mathrm{Fin}\, X$ holds $\bigcap F \in \mathrm{Fin}\, X$.

Let X be a non empty set. Let us observe that X is trivial if and only if:

(Def. 1) For all elements x, y of X holds $x = y$.

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2. Families of complements

We now state a number of propositions:

(3) For every set X and for every family F of subsets of X and for every subset P of X holds $P^{c} \in F^{c}$ iff $P \in F$.

(4) For every set X and for every family F of subsets of X holds $F \approx F^{c}$.

(5) For all sets X, Y such that $X \approx Y$ and X is countable holds Y is countable.

(6) For every set X and for every family F of subsets of X holds $(F^{c})^{c} = F$.

(7) For every set X and for every family F of subsets of X and for every subset P of X holds $P^{c} \in F^{c}$ iff $P \in F$.

(8) For every set X and for all families F, G of subsets of X such that $F^{c} \subseteq G^{c}$ holds $F \subseteq G$.

(9) For every set X and for all families F, G of subsets of X holds $F^{c} \subseteq G$ iff $F \subseteq G^{c}$.

(10) For every set X and for all families F, G of subsets of X such that $F^{c} = G^{c}$ holds $F = G$.

(11) For every set X and for all families F, G of subsets of X holds $(F \cup G)^{c} = F^{c} \cup G^{c}$.

(12) For every set X and for every family F of subsets of X such that $F = \{X\}$ holds $F^{c} = \{\emptyset\}$.

Let X be a set and let F be an empty family of subsets of X. Observe that F^{c} is empty.

The following propositions are true:

(13) Let X be a 1-sorted structure, F be a family of subsets of X, and P be a subset of the carrier of X. Then $P \in F^{c}$ if and only if $-P \in F$.

(14) Let X be a 1-sorted structure, F be a family of subsets of X, and P be a subset of the carrier of X. Then $-P \in F^{c}$ if and only if $P \in F$.

(15) For every 1-sorted structure X and for every family F of subsets of X holds $\mathrm{Intersect}(F^{c}) = -\bigcup F$.

(16) For every 1-sorted structure X and for every family F of subsets of X holds $\bigcup(F^{c}) = -\mathrm{Intersect}(F)$.

3. Topological preliminaries

One can prove the following four propositions:

(17) Let T be a non empty topological space and A, B be subsets of the carrier of T. Suppose $B \subseteq A$ and A is closed and for every subset C of the carrier of T such that $B \subseteq C$ and C is closed holds $A \subseteq C$. Then $A = \overline{B}$.

(18) Let T be a topological structure, B be a basis of T, and V be a subset of T. If V is open, then $V = \bigcup\{G, G$ ranges over subsets of T: $G \in B \wedge G \subseteq V\}$.

(19) Let T be a topological structure, B be a basis of T, and S be a subset of T. If $S \in B$, then S is open.

(20) Let T be a non empty topological space, B be a basis of T, and V be a subset of T. Then $\operatorname{Int} V = \bigcup\{G, G$ ranges over subsets of T: $G \in B \wedge G \subseteq V\}$.

4. Baire Spaces

Let T be a non empty topological structure and let x be a point of T. A family of subsets of T is called a basis of x if it satisfies the conditions (Def. 2).

(Def. 2)(i) It $\subseteq$ the topology of T,

(ii) $x \in$ Intersect(it), and

(iii) for every subset S of T such that S is open and $x \in S$ there exists a subset V of T such that $V \in$ it and $V \subseteq S$.

Next we state three propositions:

(21) Let T be a non empty topological structure, x be a point of T, B be a basis of x, and V be a subset of T. If $V \in B$, then V is open and $x \in V$.

(22) Let T be a non empty topological structure, x be a point of T, B be a basis of x, and V be a subset of the carrier of T. If $x \in V$ and V is open, then there exists a subset W of T such that $W \in B$ and $W \subseteq V$.

(23) Let T be a non empty topological structure and P be a family of subsets of T. Suppose $P \subseteq$ the topology of T and for every point x of T there exists a basis B of x such that $B \subseteq P$. Then P is a basis of T.

Let T be a non empty topological space. We say that T is Baire if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let F be a family of subsets of T. Suppose F is countable and for every subset S of T such that $S \in F$ holds S is everywhere dense. Then Intersect(F) is dense.

We now state the proposition

(24) Let T be a non empty topological space. Then T is Baire if and only if for every family F of subsets of T such that F is countable and for every subset S of T such that $S \in F$ holds S is nowhere dense holds $\bigcup F$ is boundary.

5. Sober Spaces

Let T be a topological structure and let S be a subset of T. We say that S is irreducible if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) S is non empty and closed, and

(ii) for all subsets S_1, S_2 of T such that S_1 is closed and S_2 is closed and $S = S_1 \cup S_2$ holds $S_1 = S$ or $S_2 = S$.

Let T be a topological structure. Observe that every subset of T which is irreducible is also non empty.

Let T be a non empty topological space, let S be a subset of the carrier of T, and let p be a point of T. We say that p is dense point of S if and only if:

(Def. 5) $p \in S$ and $S \subseteq \overline{\{p\}}$.

We now state two propositions:

(25) Let T be a non empty topological space and S be a subset of the carrier of T. Suppose S is closed. Let p be a point of T. If p is dense point of S, then $S = \overline{\{p\}}$.

(26) For every non empty topological space T and for every point p of T holds $\overline{\{p\}}$ is irreducible.

Let T be a non empty topological space. Observe that there exists a subset of T which is irreducible.

Let T be a non empty topological space. We say that T is sober if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let S be an irreducible subset of T. Then there exists a point p of T such that p is dense point of S and for every point q of T such that q is dense point of S holds $p = q$.

We now state four propositions:

(27) For every non empty topological space T and for every point p of T holds p is dense point of $\overline{\{p\}}$.

(28) For every non empty topological space T and for every point p of T holds p is dense point of $\{p\}$.

(29) Let T be a non empty topological space and G, F be subsets of T. If G is open and F is closed, then $F \setminus G$ is closed.

(30) For every Hausdorff non empty topological space T holds every irreducible subset of T is trivial.

Let T be a Hausdorff non empty topological space. Observe that every subset of T which is irreducible is also trivial.

We now state the proposition

(31) Every Hausdorff non empty topological space is sober.

Let us note that every non empty topological space which is Hausdorff is also sober.

One can verify that there exists a non empty topological space which is sober.

The following two propositions are true:

(32) Let T be a non empty topological space. Then T is T_0 if and only if for all points p, q of T such that $\overline{\{p\}} = \overline{\{q\}}$ holds $p = q$.

(33) Every sober non empty topological space is T_0.

Let us note that every non empty topological space which is sober is also T_0.

Let X be a set. The functor CofinTop X yields a strict topological structure and is defined as follows:

(Def. 7) The carrier of CofinTop $X = X$ and (the topology of CofinTop $X)^{\rm c} = \{X\} \cup {\rm Fin}\, X$.

Let X be a non empty set. Note that CofinTop X is non empty.

Let X be a set. Note that CofinTop X is topological space-like.

Next we state two propositions:

(34) For every non empty set X and for every subset P of CofinTop X holds P is closed iff $P = X$ or P is finite.

(35) For every non empty topological space T such that T is a T_1 space and for every point p of T holds $\overline{\{p\}} = \{p\}$.

Let X be a non empty set. Note that CofinTop X is a T_1 space.

Let X be an infinite set. One can check that CofinTop X is non sober.

Let us observe that there exists a non empty topological space which is a T_1 space and non sober.

6. More on regular spaces

One can prove the following two propositions:

(36) Let T be a non empty topological space. Then T is a T_3 space if and only if for every point p of T and for every subset P of the carrier of T such that $p \in {\rm Int}\, P$ there exists a subset Q of T such that Q is closed and $Q \subseteq P$ and $p \in {\rm Int}\, Q$.

(37) Let T be a non empty topological space. Suppose T is a T_3 space. Then T is locally-compact if and only if for every point x of T there exists a subset Y of T such that $x \in {\rm Int}\, Y$ and Y is compact.

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