FORMALIZED MATHEMATICS
Volume 6, Number 3, 1997
Warsaw University - Białystok

Basic Properties of Objects and Morphisms

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MML Identifier: ALTCAT_3.

The articles [7], [9], [10], [1], [3], [4], [2], [8], [6], and [5] provide the notation and terminology for this paper.

Let C be a non empty category structure with units, let o_1, o_2 be objects of C, let A be a morphism from o_1 to o_2, and let B be a morphism from o_2 to o_1. We say that A is left inverse of B if and only if:

(Def. 1) $A \cdot B = \mathrm{id}_{(o_2)}$.

We introduce B is right inverse of A as a synonym of A is left inverse of B.

Let C be a non empty category structure with units, let o_1, o_2 be objects of C, and let A be a morphism from o_1 to o_2. We say that A is retraction if and only if:

(Def. 2) There exists a morphism from o_2 to o_1 which is right inverse of A.

Let C be a non empty category structure with units, let o_1, o_2 be objects of C, and let A be a morphism from o_1 to o_2. We say that A is coretraction if and only if:

(Def. 3) There exists a morphism from o_2 to o_1 which is left inverse of A.

Next we state the proposition

(1) Let C be a non empty category structure with units and o be an object of C. Then id_o is retraction and id_o is coretraction.

Let C be a category and let o_1, o_2 be objects of C. Let us assume that $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. Let us assume that A is retraction and coretraction. The functor A^{-1} yields a morphism from o_2 to o_1 and is defined by:

(Def. 4) A^{-1} is left inverse of A and A^{-1} is right inverse of A.

We now state three propositions:

ISSN 1426–2630

(2) Let C be a category and o_1, o_2 be objects of C. Suppose $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. If A is retraction and coretraction, then $A^{-1} \cdot A = \mathrm{id}_{(o_1)}$ and $A \cdot A^{-1} = \mathrm{id}_{(o_2)}$.

(3) Let C be a category and o_1, o_2 be objects of C. Suppose $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. If A is retraction and coretraction, then $(A^{-1})^{-1} = A$.

(4) For every category C and for every object o of C holds $(\mathrm{id}_o)^{-1} = \mathrm{id}_o$.

Let C be a category, let o_1, o_2 be objects of C, and let A be a morphism from o_1 to o_2. We say that A is iso if and only if:

(Def. 5) $A \cdot A^{-1} = \mathrm{id}_{(o_2)}$ and $A^{-1} \cdot A = \mathrm{id}_{(o_1)}$.

One can prove the following three propositions:

(5) Let C be a category, o_1, o_2 be objects of C, and A be a morphism from o_1 to o_2. If A is iso, then A is retraction and coretraction.

(6) Let C be a category and o_1, o_2 be objects of C. Suppose $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. Then A is iso if and only if A is retraction and coretraction.

(7) Let C be a category, o_1, o_2, o_3 be objects of C, A be a morphism from o_1 to o_2, and B be a morphism from o_2 to o_3. Suppose $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_3\rangle \neq \emptyset$ and $\langle o_3, o_1\rangle \neq \emptyset$ and A is iso and B is iso. Then $B \cdot A$ is iso and $(B \cdot A)^{-1} = A^{-1} \cdot B^{-1}$.

Let C be a category and let o_1, o_2 be objects of C. We say that o_1, o_2 are iso if and only if:

(Def. 6) $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$ and there exists a morphism from o_1 to o_2 which is iso.

Let us note that the predicate o_1, o_2 are iso is reflexive and symmetric.

One can prove the following proposition

(8) Let C be a category and o_1, o_2, o_3 be objects of C. If o_1, o_2 are iso and o_2, o_3 are iso , then o_1, o_3 are iso .

Let C be a non empty category structure, let o_1, o_2 be objects of C, and let A be a morphism from o_1 to o_2. We say that A is mono if and only if:

(Def. 7) For every object o of C such that $\langle o, o_1\rangle \neq \emptyset$ and for all morphisms B, C from o to o_1 such that $A \cdot B = A \cdot C$ holds $B = C$.

Let C be a non empty category structure, let o_1, o_2 be objects of C, and let A be a morphism from o_1 to o_2. We say that A is epi if and only if:

(Def. 8) For every object o of C such that $\langle o_2, o\rangle \neq \emptyset$ and for all morphisms B, C from o_2 to o such that $B \cdot A = C \cdot A$ holds $B = C$.

We now state a number of propositions:

(9) Let C be an associative transitive non empty category structure and o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_3\rangle \neq \emptyset$. Let A be a

morphism from o_1 to o_2 and B be a morphism from o_2 to o_3. If A is mono and B is mono, then $B \cdot A$ is mono.

(10) Let C be an associative transitive non empty category structure and o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and B be a morphism from o_2 to o_3. If A is epi and B is epi, then $B \cdot A$ is epi.

(11) Let C be an associative transitive non empty category structure and o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and B be a morphism from o_2 to o_3. If $B \cdot A$ is mono, then A is mono.

(12) Let C be an associative transitive non empty category structure and o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and B be a morphism from o_2 to o_3. If $B \cdot A$ is epi, then B is epi.

(13) Let X be a non empty set and o_1, o_2 be objects of Ens_X. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and F be a function from o_1 into o_2. If $F = A$, then A is mono iff F is one-to-one.

(14) Let X be a non empty set with non empty elements and o_1, o_2 be objects of Ens_X. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and F be a function from o_1 into o_2. If $F = A$, then A is epi iff F is onto.

(15) Let C be a category and o_1, o_2 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. If A is retraction, then A is epi.

(16) Let C be a category and o_1, o_2 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. If A is coretraction, then A is mono.

(17) Let C be a category and o_1, o_2 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. If A is iso, then A is mono and epi.

(18) Let C be a category and o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_3, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and B be a morphism from o_2 to o_3. If A is retraction and B is retraction, then $B \cdot A$ is retraction.

(19) Let C be a category and o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_3, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and B be a morphism from o_2 to o_3. If A is coretraction and B is coretraction, then $B \cdot A$ is coretraction.

(20) Let C be a category, o_1, o_2 be objects of C, and A be a morphism from o_1 to o_2. If A is retraction and mono and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$, then A is iso.

(21) Let C be a category, o_1, o_2 be objects of C, and A be a morphism from o_1 to o_2. If A is coretraction and epi and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$, then A is iso.

(22) Let C be a category, o_1, o_2, o_3 be objects of C, A be a morphism from o_1 to o_2, and B be a morphism from o_2 to o_3. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_3, o_1 \rangle \neq \emptyset$ and $B \cdot A$ is retraction. Then B is retraction.

(23) Let C be a category, o_1, o_2, o_3 be objects of C, A be a morphism from o_1 to o_2, and B be a morphism from o_2 to o_3. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_3, o_1 \rangle \neq \emptyset$ and $B \cdot A$ is coretraction. Then A is coretraction.

(24) Let C be a category. Suppose that for all objects o_1, o_2 of C holds every morphism from o_1 to o_2 is retraction. Let a, b be objects of C and A be a morphism from a to b. If $\langle a, b \rangle \neq \emptyset$ and $\langle b, a \rangle \neq \emptyset$, then A is iso.

Let C be a non empty category structure with units and let o be an object of C. Note that there exists a morphism from o to o which is mono, epi, retraction, and coretraction.

Let C be a category and let o be an object of C. Observe that there exists a morphism from o to o which is mono, epi, iso, retraction, and coretraction.

Let C be a category, let o be an object of C, and let A, B be mono morphisms from o to o. Note that $A \cdot B$ is mono.

Let C be a category, let o be an object of C, and let A, B be epi morphisms from o to o. Observe that $A \cdot B$ is epi.

Let C be a category, let o be an object of C, and let A, B be iso morphisms from o to o. One can verify that $A \cdot B$ is iso.

Let C be a category, let o be an object of C, and let A, B be retraction morphisms from o to o. Observe that $A \cdot B$ is retraction.

Let C be a category, let o be an object of C, and let A, B be coretraction morphisms from o to o. One can check that $A \cdot B$ is coretraction.

Let C be a graph and let o be an object of C. We say that o is initial if and only if:

(Def. 9) For every object o_1 of C there exists a morphism M from o to o_1 such that $M \in \langle o, o_1 \rangle$ and $\langle o, o_1 \rangle$ is trivial.

One can prove the following two propositions:

(25) Let C be a graph and o be an object of C. Then o is initial if and only if for every object o_1 of C there exists a morphism M from o to o_1 such that $M \in \langle o, o_1 \rangle$ and for every morphism M_1 from o to o_1 such that $M_1 \in \langle o, o_1 \rangle$ holds $M = M_1$.

(26) For every category C and for all objects o_1, o_2 of C such that o_1 is initial and o_2 is initial holds o_1, o_2 are iso .

Let C be a graph and let o be an object of C. We say that o is terminal if and only if:

(Def. 10) For every object o_1 of C there exists a morphism M from o_1 to o such that $M \in \langle o_1, o \rangle$ and $\langle o_1, o \rangle$ is trivial.

Next we state two propositions:

(27) Let C be a graph and o be an object of C. Then o is terminal if and only if for every object o_1 of C there exists a morphism M from o_1 to o such that $M \in \langle o_1, o \rangle$ and for every morphism M_1 from o_1 to o such that $M_1 \in \langle o_1, o \rangle$ holds $M = M_1$.

(28) For every category C and for all objects o_1, o_2 of C such that o_1 is terminal and o_2 is terminal holds o_1, o_2 are iso .

Let C be a graph and let o be an object of C. We say that o is zero if and only if:

(Def. 11) o is initial and terminal.

We now state the proposition

(29) For every category C and for all objects o_1, o_2 of C such that o_1 is zero and o_2 is zero holds o_1, o_2 are iso .

Let C be a non empty category structure, let o_1, o_2 be objects of C, and let M be a morphism from o_1 to o_2. We say that M is zero if and only if the condition (Def. 12) is satisfied.

(Def. 12) Let o be an object of C. Suppose o is zero. Let A be a morphism from o_1 to o and B be a morphism from o to o_2. Then $M = B \cdot A$.

We now state the proposition

(30) Let C be a category, o_1, o_2, o_3 be objects of C, M_1 be a morphism from o_1 to o_2, and M_2 be a morphism from o_2 to o_3. If M_1 is zero and M_2 is zero, then $M_2 \cdot M_1$ is zero.

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Received February 14, 1997