FORMALIZED MATHEMATICS
Volume 6, Number 3, 1997
Warsaw University - Białystok

Equations in Many Sorted Algebras

Artur Korniłowicz
Warsaw University
Białystok

Summary. This paper is preparation to prove Birkhoff's Theorem. Some properties of many sorted algebras are proved. The last section of this work shows that every equation valid in a many sorted algebra is also valid in each subalgebra, and each image of it. Moreover for a family of many sorted algebras $(A_i : i \in I)$ if every equation is valid in each A_i, $i \in I$ then is also valid in product $\prod(A_i : i \in I)$.

MML Identifier: `EQUATION`.

The articles [23], [28], [10], [29], [6], [9], [7], [24], [11], [4], [8], [1], [2], [25], [26], [18], [19], [27], [20], [5], [12], [16], [17], [13], [22], [21], [15], [14], and [3] provide the notation and terminology for this paper.

1. On the Functions and Many Sorted Functions

In this paper I is a set.

Next we state several propositions:

(1) Let A be a set, B, C be non empty sets, f be a function from A into B, and g be a function from B into C. If $\operatorname{rng}(g \cdot f) = C$, then $\operatorname{rng} g = C$.

(2) Let A be a many sorted set indexed by I, B, C be non-empty many sorted sets indexed by I, f be a many sorted function from A into B, and g be a many sorted function from B into C. If $g \circ f$ is onto, then g is onto.

(3) Let A, B be non empty sets, C, y be sets, and f be a function. If $f \in (C^B)^A$ and $y \in B$, then $\operatorname{dom}(\operatorname{commute}(f))(y) = A$ and $\operatorname{rng}(\operatorname{commute}(f))(y) \subseteq C$.

ISSN 1426–2630

(4) For every many sorted set A indexed by I there exists a non-empty many sorted set B indexed by I such that $A \subseteq B$.

(5) Let A, B be many sorted sets indexed by I. Suppose A is transformable to B. Let f be a many sorted function indexed by I. If $\mathrm{dom}_\kappa f(\kappa) = A$ and $\mathrm{rng}_\kappa f(\kappa) \subseteq B$, then f is a many sorted function from A into B.

(6) Let A, B be many sorted sets indexed by I, F be a many sorted function from A into B, C, E be many sorted subsets indexed by A, and D be a many sorted subset indexed by C. If $E = D$, then $F \restriction C \restriction D = F \restriction E$.

(7) Let B be a non-empty many sorted set indexed by I, C be a many sorted set indexed by I, A be a many sorted subset indexed by C, and F be a many sorted function from A into B. Then there exists a many sorted function G from C into B such that $G \restriction A = F$.

Let I be a set, let A be a many sorted set indexed by I, and let F be a many sorted function indexed by I. The functor $F^{-1}(A)$ yielding a many sorted set indexed by I is defined as follows:

(Def. 1) For every set i such that $i \in I$ holds $(F^{-1}(A))(i) = F(i)^{-1}(A(i))$.

We now state a number of propositions:

(8) Let A, B, C be many sorted sets indexed by I and F be a many sorted function from A into B. Then $F \circ C$ is a many sorted subset indexed by B.

(9) Let A, B, C be many sorted sets indexed by I and F be a many sorted function from A into B. Then $F^{-1}(C)$ is a many sorted subset indexed by A.

(10) Let A, B be many sorted sets indexed by I and F be a many sorted function from A into B. If F is onto, then $F \circ A = B$.

(11) Let A, B be many sorted sets indexed by I and F be a many sorted function from A into B. If A is transformable to B, then $F^{-1}(B) = A$.

(12) Let A be a many sorted set indexed by I and F be a many sorted function indexed by I. If $A \subseteq \mathrm{rng}_\kappa F(\kappa)$, then $F \circ F^{-1}(A) = A$.

(13) For every many sorted function f indexed by I and for every many sorted set X indexed by I holds $f \circ X \subseteq \mathrm{rng}_\kappa f(\kappa)$.

(14) For every many sorted function f indexed by I holds $f \circ (\mathrm{dom}_\kappa f(\kappa)) = \mathrm{rng}_\kappa f(\kappa)$.

(15) For every many sorted function f indexed by I holds $f^{-1}(\mathrm{rng}_\kappa f(\kappa)) = \mathrm{dom}_\kappa f(\kappa)$.

(16) For every many sorted set A indexed by I holds $(\mathrm{id}_A) \circ A = A$.

(17) For every many sorted set A indexed by I holds $(\mathrm{id}_A)^{-1}(A) = A$.

2. On the Many Sorted Algebras

In the sequel S denotes a non empty non void many sorted signature and U_0, U_1 denote non-empty algebras over S.

One can prove the following propositions:

(18) For every algebra A over S holds the algebra of A is a subalgebra of A.

(19) Every algebra A over S is a subalgebra of the algebra of A.

(20) Let U_0 be an algebra over S, A be a subalgebra of U_0, o be an operation symbol of S, and x be a set. If $x \in \mathrm{Args}(o, A)$, then $x \in \mathrm{Args}(o, U_0)$.

(21) Let U_0 be an algebra over S, A be a subalgebra of U_0, o be an operation symbol of S, and x be a set. If $x \in \mathrm{Args}(o, A)$, then $(\mathrm{Den}(o, A))(x) = (\mathrm{Den}(o, U_0))(x)$.

(22) Let F be an algebra family of I over S, B be a subalgebra of $\prod F$, o be an operation symbol of S, and x be a set. If $x \in \mathrm{Args}(o, B)$, then $(\mathrm{Den}(o, B))(x)$ is a function and $(\mathrm{Den}(o, \prod F))(x)$ is a function.

Let S be a non void non empty many sorted signature, let A be an algebra over S, and let B be a subalgebra of A. The functor SuperAlgebraSet(B) is defined by the condition (Def. 2).

(Def. 2) Let x be a set. Then $x \in$ SuperAlgebraSet(B) if and only if there exists a strict subalgebra C of A such that $x = C$ and B is a subalgebra of C.

Let S be a non void non empty many sorted signature, let A be an algebra over S, and let B be a subalgebra of A. Note that SuperAlgebraSet(B) is non empty.

Let S be a non empty non void many sorted signature. One can verify that there exists an algebra over S which is strict, non-empty, and free.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let X be a non-empty locally-finite subset of A. One can verify that Gen(X) is finitely-generated.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S. Note that there exists a subalgebra of A which is strict, non-empty, and finitely-generated.

Let S be a non empty non void many sorted signature and let A be a feasible algebra over S. Note that there exists a subalgebra of A which is feasible.

Next we state several propositions:

(23) Let A be an algebra over S, C be a subalgebra of A, and D be a many sorted subset indexed by the sorts of A. Suppose $D =$ the sorts of C. Let h be a many sorted function from A into U_0 and g be a many sorted function from C into U_0. Suppose $g = h \restriction D$. Let o be an operation symbol of S, x be an element of $\mathrm{Args}(o, A)$, and y be an element of $\mathrm{Args}(o, C)$. If $\mathrm{Args}(o, C) \neq \emptyset$ and $x = y$, then $h \# x = g \# y$.

(24) Let A be a feasible algebra over S, C be a feasible subalgebra of A, and D be a many sorted subset indexed by the sorts of A. Suppose $D =$ the sorts of C. Let h be a many sorted function from A into U_0. Suppose h is a homomorphism of A into U_0. Let g be a many sorted function from C into U_0. If $g = h \restriction D$, then g is a homomorphism of C into U_0.

(25) Let B be a strict non-empty algebra over S, G be a generator set of U_0, H be a non-empty generator set of B, and f be a many sorted function from U_0 into B. Suppose $H \subseteq f \circ G$ and f is a homomorphism of U_0 into B. Then f is an epimorphism of U_0 onto B.

(26) Let W be a strict free non-empty algebra over S and F be a many sorted function from U_0 into U_1. Suppose F is an epimorphism of U_0 onto U_1. Let G be a many sorted function from W into U_1. Suppose G is a homomorphism of W into U_1. Then there exists a many sorted function H from W into U_0 such that H is a homomorphism of W into U_0 and $G = F \circ H$.

(27) Let I be a non empty finite set, A be a non-empty algebra over S, and F be an algebra family of I over S. Suppose that for every element i of I there exists a strict non-empty finitely-generated subalgebra C of A such that $C = F(i)$. Then there exists a strict non-empty finitely-generated subalgebra B of A such that for every element i of I holds $F(i)$ is a subalgebra of B.

(28) Let A, B be strict non-empty finitely-generated subalgebras of U_0. Then there exists a strict non-empty finitely-generated subalgebra M of U_0 such that A is a subalgebra of M and B is a subalgebra of M.

(29) Let S_1 be a non empty non void many sorted signature, A_1 be a non-empty algebra over S_1, and C be a set. Suppose $C = \{A, A$ ranges over elements of Subalgebras(A_1): $\bigvee_{R:\ \text{strict non-empty finitely-generated subalgebra of}\ A_1} R = A\}$. Let F be an algebra family of C over S_1. Suppose that for every set c such that $c \in C$ holds $c = F(c)$. Then there exists a strict non-empty subalgebra P_1 of $\prod F$ such that there exists a many sorted function from P_1 into A_1 which is an epimorphism of P_1 onto A_1.

(30) Let U_0 be a feasible free algebra over S, A be a free generator set of U_0, and Z be a subset of U_0. If $Z \subseteq A$ and Gen(Z) is feasible, then Gen(Z) is free.

3. Equations in Many Sorted Algebras

Let S be a non empty non void many sorted signature. The functor $\mathrm{T}_S(\mathbb{N})$ yielding an algebra over S is defined by:

(Def. 3) $\mathrm{T}_S(\mathbb{N}) = \mathrm{Free}((\text{the carrier of } S) \longmapsto \mathbb{N})$.

Let S be a non empty non void many sorted signature. Note that $\mathrm{T}_S(\mathbb{N})$ is strict non-empty and free.

Let S be a non empty non void many sorted signature. The equations of S constitute a many sorted set indexed by the carrier of S and is defined by:

(Def. 4) The equations of $S = [\![\text{the sorts of } \mathrm{T}_S(\mathbb{N}), \text{the sorts of } \mathrm{T}_S(\mathbb{N})]\!]$.

Let S be a non empty non void many sorted signature. Observe that the equations of S is non-empty.

Let S be a non empty non void many sorted signature. A set of equations of S is a many sorted subset indexed by the equations of S.

In the sequel s denotes a sort symbol of S, e denotes an element of (the equations of S)(s), and E denotes a set of equations of S.

Let S be a non empty non void many sorted signature, let s be a sort symbol of S, and let x, y be elements of (the sorts of $\mathrm{T}_S(\mathbb{N})$)(s). Then $\langle x, y\rangle$ is an element of (the equations of S)(s). We introduce x=y as a synonym of $\langle x, y\rangle$.

Next we state two propositions:

(31) $e_{\mathbf{1}} \in (\text{the sorts of } \mathrm{T}_S(\mathbb{N}))(s)$.

(32) $e_{\mathbf{2}} \in (\text{the sorts of } \mathrm{T}_S(\mathbb{N}))(s)$.

Let S be a non empty non void many sorted signature, let A be an algebra over S, let s be a sort symbol of S, and let e be an element of (the equations of S)(s). The predicate $A \models e$ is defined by:

(Def. 5) For every many sorted function h from $\mathrm{T}_S(\mathbb{N})$ into A such that h is a homomorphism of $\mathrm{T}_S(\mathbb{N})$ into A holds $h(s)(e_{\mathbf{1}}) = h(s)(e_{\mathbf{2}})$.

Let S be a non empty non void many sorted signature, let A be an algebra over S, and let E be a set of equations of S. The predicate $A \models E$ is defined as follows:

(Def. 6) For every sort symbol s of S and for every element e of (the equations of S)(s) such that $e \in E(s)$ holds $A \models e$.

We now state several propositions:

(33) For every strict non-empty subalgebra U_2 of U_0 such that $U_0 \models e$ holds $U_2 \models e$.

(34) For every strict non-empty subalgebra U_2 of U_0 such that $U_0 \models E$ holds $U_2 \models E$.

(35) If U_0 and U_1 are isomorphic and $U_0 \models e$, then $U_1 \models e$.

(36) If U_0 and U_1 are isomorphic and $U_0 \models E$, then $U_1 \models E$.

(37) For every congruence R of U_0 such that $U_0 \models e$ holds $U_0/R \models e$.

(38) For every congruence R of U_0 such that $U_0 \models E$ holds $U_0/R \models E$.

(39) Let F be an algebra family of I over S. Suppose that for every set i such that $i \in I$ there exists an algebra A over S such that $A = F(i)$ and $A \models e$. Then $\prod F \models e$.

(40) Let F be an algebra family of I over S. Suppose that for every set i such that $i \in I$ there exists an algebra A over S such that $A = F(i)$ and $A \models E$. Then $\prod F \models E$.

References

[1] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.

[2] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.

[3] Grzegorz Bancerek. Translations, endomorphisms, and stable equational theories. *Formalized Mathematics*, 5(**4**):553–564, 1996.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[5] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Formalized Mathematics*, 5(**1**):47–54, 1996.

[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[8] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[11] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[12] Mariusz Giero. More on products of many sorted algebras. *Formalized Mathematics*, 5(**4**):621–626, 1996.

[13] Artur Korniłowicz. Extensions of mappings on generator set. *Formalized Mathematics*, 5(**2**):269–272, 1996.

[14] Artur Korniłowicz. On the closure operator and the closure system of many sorted sets. *Formalized Mathematics*, 5(**4**):543–551, 1996.

[15] Artur Korniłowicz. On the group of automorphisms of universal algebra & many sorted algebra. *Formalized Mathematics*, 5(**2**):221–226, 1996.

[16] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Formalized Mathematics*, 5(**1**):61–65, 1996.

[17] Małgorzata Korolkiewicz. Many sorted quotient algebra. *Formalized Mathematics*, 5(**1**):79–84, 1996.

[18] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.

[19] Beata Madras. Products of many sorted algebras. *Formalized Mathematics*, 5(**1**):55–60, 1996.

[20] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Formalized Mathematics*, 5(**2**):167–172, 1996.

[21] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, II. *Formalized Mathematics*, 5(**2**):215–220, 1996.

[22] Beata Perkowska. Free many sorted universal algebra. *Formalized Mathematics*, 5(**1**):67–74, 1996.

[23] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[24] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[25] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.

[26] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.

[27] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[28] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[29] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received May 30, 1997