FORMALIZED MATHEMATICS
Volume 6, Number 3, 1997
Warsaw University - Białystok

Subsequences of Standard Special Circular Sequences in $\mathcal{E}^2_{\rm T}$

Yatsuka Nakamura
Shinshu University
Nagano

Roman Matuszewski[1]
Warsaw University
Białystok

Adam Grabowski[2]
Warsaw University
Białystok

Summary. It is known that a standard special circular sequence in $\mathcal{E}^2_{\rm T}$ properly defines a special polygon. We are interested in a part of such a sequence. It is shown that if the first point and the last point of the subsequence are different, it becomes a special polygonal sequence. The concept of „a part of" is introduced, and the subsequence having this property can be characterized by using „mid" function. For such subsequences, the concepts of „Upper" and „Lower" parts are introduced.

MML Identifier: `JORDAN4`.

The notation and terminology used here are introduced in the following papers: [16], [19], [8], [1], [14], [20], [2], [3], [18], [4], [6], [7], [11], [10], [13], [15], [5], [17], [9], and [12].

1. Preliminaries

We adopt the following convention: i, i_1, i_2, i_3, j, k, n denote natural numbers and r_1, r_2, s, s_1 denote real numbers.

The following propositions are true:

(1) If $n -' i = 0$, then $n \leqslant i$.

(2) If $i \leqslant j$, then $(j + k) -' i = (j + k) - i$.

[1]This paper was written while the author visited Shinshu University in fall 1996.
[2]This paper was written while the author visited Shinshu University in winter 1997.

ISSN 1426–2630

(3) If $i \leqslant j$, then $(j + k) -' i = j -' i + k$.
(4) If $i_1 \neq 0$ and $i_2 = i_3 \cdot i_1$, then $i_3 \leqslant i_2$.
(5) If $i_1 < i_2$, then $i_1 \div i_2 = 0$.
(6) If $0 < j$ and $j < i$ and $i < j + j$, then $i \bmod j \neq 0$.
(7) If $0 < j$ and $j \leqslant i$ and $i < j + j$, then $i \bmod j = i - j$ and $i \bmod j = i -' j$.
(8) If $0 < j$, then $(j + j) \bmod j = 0$ and $k \cdot j \bmod j = 0$.
(9) If $0 < k$ and $k \leqslant j$ and $k \bmod j = 0$, then $k = j$.
(10) $(r_1 + s_1 + r_2) - s_1 = r_1 + r_2$ and $(r_1 - s_1) + r_2 + s_1 = r_1 + r_2$ and $(r_1 + s_1) - r_2 - s_1 = r_1 - r_2$ and $(r_1 - s_1 - r_2) + s_1 = r_1 - r_2$.
(11) $r_1 - r_1 - r_2 = -r_2$ and $(-r_1 + r_1) - r_2 = -r_2$ and $r_1 - r_2 - r_1 = -r_2$ and $(-r_1 - r_2) + r_1 = -r_2$.
(12) If $0 < s$ and if $s \cdot r_1 \leqslant s \cdot r_2$ or $r_1 \cdot s \leqslant r_2 \cdot s$, then $r_1 \leqslant r_2$.
(13) If $0 < s$ and if $s \cdot r_1 < s \cdot r_2$ or $r_1 \cdot s < r_2 \cdot s$, then $r_1 < r_2$.

2. Some facts about cutting of finite sequences

In the sequel D denotes a non empty set, f_1 denotes a finite sequence of elements of D, and f denotes a non constant standard special circular sequence.

We now state a number of propositions:

(14) For every f_1 such that f_1 is circular and $1 \leqslant \operatorname{len} f_1$ holds $f_1(1) = f_1(\operatorname{len} f_1)$.
(15) For all f_1, i_1, i_2 such that $i_1 \leqslant i_2$ holds $f_1 \restriction i_1 \restriction i_2 = f_1 \restriction i_1$ and $f_1 \restriction i_2 \restriction i_1 = f_1 \restriction i_1$.
(16) $\varepsilon_D \restriction i = \varepsilon_D$.
(17) $\operatorname{Rev}(\varepsilon_D) = \varepsilon_D$.
(18) For all f_1, k such that $k < \operatorname{len} f_1$ holds $(f_1)_{\lfloor k}(\operatorname{len}((f_1)_{\lfloor k})) = f_1(\operatorname{len} f_1)$ and $\pi_{\operatorname{len}((f_1)_{\lfloor k})}(f_1)_{\lfloor k} = \pi_{\operatorname{len} f_1} f_1$.
(19) Let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i. If g is a special sequence and $i + 1 < \operatorname{len} g$, then $g_{\lfloor i}$ is a special sequence.
(20) For all f_1, i_1, i_2 such that $1 \leqslant i_2$ and $i_2 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} f_1$ holds $\operatorname{len} \operatorname{mid}(f_1, i_2, i_1) = i_1 -' i_2 + 1$.
(21) For all f_1, i_1, i_2 such that $1 \leqslant i_2$ and $i_2 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} f_1$ holds $\operatorname{len} \operatorname{mid}(f_1, i_1, i_2) = i_1 -' i_2 + 1$.
(22) For all f_1, i_1, i_2, j such that $1 \leqslant i_1$ and $i_1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} f_1$ holds $(\operatorname{mid}(f_1, i_1, i_2))(\operatorname{len} \operatorname{mid}(f_1, i_1, i_2)) = f_1(i_2)$.
(23) For all f_1, i_1, i_2, j such that $1 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} f_1$ and $1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} f_1$ holds $(\operatorname{mid}(f_1, i_1, i_2))(\operatorname{len} \operatorname{mid}(f_1, i_1, i_2)) = f_1(i_2)$.

(24) For all f_1, i_1, i_2, j such that $1 \leqslant i_2$ and $i_2 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} f_1$ and $1 \leqslant j$ and $j \leqslant i_1 -' i_2 + 1$ holds $(\operatorname{mid}(f_1, i_1, i_2))(j) = f_1(i_1 -' j + 1)$.

(25) Let given f_1, i_1, i_2. Suppose $1 \leqslant i_2$ and $i_2 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} f_1$ and $1 \leqslant j$ and $j \leqslant i_1 -' i_2 + 1$. Then $(\operatorname{mid}(f_1, i_1, i_2))(j) = (\operatorname{mid}(f_1, i_2, i_1))((((i_1 - i_2) + 1) - j) + 1)$ and $(((i_1 - i_2) + 1) - j) + 1 = (i_1 -' i_2 + 1) -' j + 1$.

(26) Let given f_1, i_1, i_2. Suppose $1 \leqslant i_1$ and $i_1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} f_1$ and $1 \leqslant j$ and $j \leqslant i_2 -' i_1 + 1$. Then $(\operatorname{mid}(f_1, i_1, i_2))(j) = (\operatorname{mid}(f_1, i_2, i_1))((((i_2 - i_1) + 1) - j) + 1)$ and $(((i_2 - i_1) + 1) - j) + 1 = (i_2 -' i_1 + 1) -' j + 1$.

(27) For all f_1, k such that $1 \leqslant k$ and $k \leqslant \operatorname{len} f_1$ holds $\operatorname{mid}(f_1, k, k) = \langle \pi_k f_1 \rangle$ and $\operatorname{len} \operatorname{mid}(f_1, k, k) = 1$.

(28) $\operatorname{mid}(f_1, 0, 0) = f_1 \restriction 1$.

(29) For all f_1, k such that $\operatorname{len} f_1 < k$ holds $\operatorname{mid}(f_1, k, k) = \varepsilon_D$.

(30) For all f_1, i_1, i_2 holds $\operatorname{mid}(f_1, i_1, i_2) = \operatorname{Rev}(\operatorname{mid}(f_1, i_2, i_1))$.

(31) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i_1, i_2, i. If $1 \leqslant i_1$ and $i_1 < i_2$ and $i_2 \leqslant \operatorname{len} f$ and $1 \leqslant i$ and $i < i_2 -' i_1 + 1$, then $\mathcal{L}(\operatorname{mid}(f, i_1, i_2), i) = \mathcal{L}(f, (i + i_1) -' 1)$.

(32) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i_1, i_2, i. If $1 \leqslant i_1$ and $i_1 < i_2$ and $i_2 \leqslant \operatorname{len} f$ and $1 \leqslant i$ and $i < i_2 -' i_1 + 1$, then $\mathcal{L}(\operatorname{mid}(f, i_2, i_1), i) = \mathcal{L}(f, i_2 -' i)$.

3. Dividing of special circular sequences into parts

Let n be a natural number and let f be a finite sequence. The functor S_Drop(n, f) yields a natural number and is defined by:

(Def. 1) $$\text{S\_Drop}(n, f) = \begin{cases} n \bmod \operatorname{len} f -' 1, & \text{if } n \bmod \operatorname{len} f -' 1 \neq 0, \\ \operatorname{len} f -' 1, & \text{otherwise.} \end{cases}$$

Next we state three propositions:

(33) For every finite sequence f such that $0 < \operatorname{len} f -' 1$ holds S_Drop$(\operatorname{len} f -' 1, f) = \operatorname{len} f -' 1$.

(34) For every natural number n and for every finite sequence f such that $1 \leqslant n$ and $n \leqslant \operatorname{len} f -' 1$ holds S_Drop$(n, f) = n$.

(35) Let n be a natural number and f be a finite sequence. If $\operatorname{len} f > 1$ or $\operatorname{len} f -' 1 > 0$, then S_Drop$(n, f) =$ S_Drop$(n + \operatorname{len} f -' 1, f)$ and S_Drop$(n, f) =$ S_Drop$(\operatorname{len} f -' 1 + n, f)$.

Let f be a non constant standard special circular sequence, let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and let i_1, i_2 be natural numbers. We say that g is a right part of f from i_1 to i_2 if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) $1 \leqslant i_1$,

(ii) $i_1 + 1 \leqslant \operatorname{len} f$,
(iii) $1 \leqslant i_2$,
(iv) $i_2 + 1 \leqslant \operatorname{len} f$,
(v) $g(\operatorname{len} g) = f(i_2)$,
(vi) $1 \leqslant \operatorname{len} g$,
(vii) $\operatorname{len} g < \operatorname{len} f$, and
(viii) for every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} g$ holds $g(i) = f(\text{S\_Drop}((i_1 + i) -' 1, f))$.

Let f be a non constant standard special circular sequence, let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and let i_1, i_2 be natural numbers. We say that g is a left part of f from i_1 to i_2 if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $1 \leqslant i_1$,
(ii) $i_1 + 1 \leqslant \operatorname{len} f$,
(iii) $1 \leqslant i_2$,
(iv) $i_2 + 1 \leqslant \operatorname{len} f$,
(v) $g(\operatorname{len} g) = f(i_2)$,
(vi) $1 \leqslant \operatorname{len} g$,
(vii) $\operatorname{len} g < \operatorname{len} f$, and
(viii) for every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} g$ holds $g(i) = f(\text{S\_Drop}((\operatorname{len} f + i_1) -' i, f))$.

Let f be a non constant standard special circular sequence, let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and let i_1, i_2 be natural numbers. We say that g is a part of f from i_1 to i_2 if and only if:

(Def. 4) g is a right part of f from i_1 to i_2 or a left part of f from i_1 to i_2.

We now state a number of propositions:

(36) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a part of f from i_1 to i_2. Then

(i) $1 \leqslant i_1$,
(ii) $i_1 + 1 \leqslant \operatorname{len} f$,
(iii) $1 \leqslant i_2$,
(iv) $i_2 + 1 \leqslant \operatorname{len} f$,
(v) $g(\operatorname{len} g) = f(i_2)$,
(vi) $1 \leqslant \operatorname{len} g$,
(vii) $\operatorname{len} g < \operatorname{len} f$, and
(viii) for every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} g$ holds $g(i) = f(\text{S\_Drop}((i_1 + i) -' 1, f))$ or for every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} g$ holds $g(i) = f(\text{S\_Drop}((\operatorname{len} f + i_1) -' i, f))$.

(37) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is

a right part of f from i_1 to i_2 and $i_1 \leqslant i_2$. Then $\operatorname{len} g = i_2 -' i_1 + 1$ and $g = \operatorname{mid}(f, i_1, i_2)$.

(38) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a right part of f from i_1 to i_2 and $i_1 > i_2$. Then $\operatorname{len} g = (\operatorname{len} f + i_2) -' i_1$ and $g = (\operatorname{mid}(f, i_1, \operatorname{len} f -' 1)) \frown (f{\restriction}i_2)$ and $g = (\operatorname{mid}(f, i_1, \operatorname{len} f -' 1)) \frown \operatorname{mid}(f, 1, i_2)$.

(39) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2 and $i_1 \geqslant i_2$. Then $\operatorname{len} g = i_1 -' i_2 + 1$ and $g = \operatorname{mid}(f, i_1, i_2)$.

(40) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2 and $i_1 < i_2$. Then $\operatorname{len} g = (\operatorname{len} f + i_1) -' i_2$ and $g = (\operatorname{mid}(f, i_1, 1)) \frown \operatorname{mid}(f, \operatorname{len} f -' 1, i_2)$.

(41) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a right part of f from i_1 to i_2. Then $\operatorname{Rev}(g)$ is a left part of f from i_2 to i_1.

(42) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2. Then $\operatorname{Rev}(g)$ is a right part of f from i_2 to i_1.

(43) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. If $1 \leqslant i_1$ and $i_1 \leqslant i_2$ and $i_2 < \operatorname{len} f$, then $\operatorname{mid}(f, i_1, i_2)$ is a right part of f from i_1 to i_2.

(44) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. If $1 \leqslant i_1$ and $i_1 \leqslant i_2$ and $i_2 < \operatorname{len} f$, then $\operatorname{mid}(f, i_2, i_1)$ is a left part of f from i_2 to i_1.

(45) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. Suppose $1 \leqslant i_2$ and $i_1 > i_2$ and $i_1 < \operatorname{len} f$. Then $(\operatorname{mid}(f, i_1, \operatorname{len} f -' 1)) \frown \operatorname{mid}(f, 1, i_2)$ is a right part of f from i_1 to i_2.

(46) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. Suppose $1 \leqslant i_1$ and $i_1 < i_2$ and $i_2 < \operatorname{len} f$. Then $(\operatorname{mid}(f, i_1, 1)) \frown \operatorname{mid}(f, \operatorname{len} f -' 1, i_2)$ is a left part of f from i_1 to i_2.

(47) Let h be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i_1, i_2. If $1 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} h$ and $1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} h$, then $\widetilde{\mathcal{L}}(\operatorname{mid}(h, i_1, i_2)) \subseteq \widetilde{\mathcal{L}}(h)$.

(48) Let g be a finite sequence of elements of D. Then g is one-to-one if and only if for all i_1, i_2 such that $1 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} g$ and $1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} g$ and $g(i_1) = g(i_2)$ or $\pi_{i_1} g = \pi_{i_2} g$ holds $i_1 = i_2$.

(49) Let f be a non constant standard special circular sequence and given i_2. If $1 < i_2$ and $i_2 + 1 \leqslant \operatorname{len} f$, then $f{\restriction}i_2$ is a special sequence.

(50) Let f be a non constant standard special circular sequence and given i_2. If $1 \leqslant i_2$ and $i_2 + 1 < \mathrm{len}\, f$, then $f_{|i_2}$ is a special sequence.

(51) Let f be a non constant standard special circular sequence and given i_1, i_2. If $1 \leqslant i_1$ and $i_1 < i_2$ and $i_2 + 1 \leqslant \mathrm{len}\, f$, then $\mathrm{mid}(f, i_1, i_2)$ is a special sequence.

(52) Let f be a non constant standard special circular sequence and given i_1, i_2. If $1 < i_1$ and $i_1 < i_2$ and $i_2 \leqslant \mathrm{len}\, f$, then $\mathrm{mid}(f, i_1, i_2)$ is a special sequence.

(53) For all points p_0, p, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_0 \in \mathcal{L}(p, q_1)$ and $p_0 \in \mathcal{L}(p, q_2)$ and $p \neq p_0$ holds $q_1 \in \mathcal{L}(p, q_2)$ or $q_2 \in \mathcal{L}(p, q_1)$.

(54) For every non constant standard special circular sequence f holds $\mathcal{L}(f, 1) \cap \mathcal{L}(f, \mathrm{len}\, f -' 1) = \{f(1)\}$.

(55) Let f be a non constant standard special circular sequence, i_1, i_2 be natural numbers, and g_1, g_2 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $1 \leqslant i_1$ and $i_1 < i_2$ and $i_2 < \mathrm{len}\, f$ and $g_1 = \mathrm{mid}(f, i_1, i_2)$ and $g_2 = (\mathrm{mid}(f, i_1, 1)) \frown \mathrm{mid}(f, \mathrm{len}\, f -' 1, i_2)$. Then $\widetilde{\mathcal{L}}(g_1) \cap \widetilde{\mathcal{L}}(g_2) = \{f(i_1), f(i_2)\}$ and $\widetilde{\mathcal{L}}(g_1) \cup \widetilde{\mathcal{L}}(g_2) = \widetilde{\mathcal{L}}(f)$.

(56) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a right part of f from i_1 to i_2 and $i_1 < i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$.

(57) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2 and $i_1 > i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$.

(58) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a right part of f from i_1 to i_2 and $i_1 \neq i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$.

(59) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2 and $i_1 \neq i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$.

(60) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a part of f from i_1 to i_2 and $i_1 \neq i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$.

(61) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a part of f from i_1 to i_2 and $g(1) \neq g(\mathrm{len}\, g)$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal

arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$.

(62) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. Suppose $1 \leqslant i_1$ and $i_1 + 1 \leqslant \operatorname{len} f$ and $1 \leqslant i_2$ and $i_2 + 1 \leqslant \operatorname{len} f$ and $i_1 \neq i_2$. Then there exist finite sequences g_1, g_2 of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that

(i) g_1 is a part of f from i_1 to i_2,

(ii) g_2 is a part of f from i_1 to i_2,

(iii) $\widetilde{\mathcal{L}}(g_1) \cap \widetilde{\mathcal{L}}(g_2) = \{f(i_1), f(i_2)\}$,

(iv) $\widetilde{\mathcal{L}}(g_1) \cup \widetilde{\mathcal{L}}(g_2) = \widetilde{\mathcal{L}}(f)$,

(v) $\widetilde{\mathcal{L}}(g_1)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$,

(vi) $\widetilde{\mathcal{L}}(g_2)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$, and

(vii) for every finite sequence g of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that g is a part of f from i_1 to i_2 holds $g = g_1$ or $g = g_2$.

In the sequel g_1, g_2 are finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$.

We now state several propositions:

(63) Let f be a non constant standard special circular sequence and P be a non empty subset of the carrier of $(\mathcal{E}^2_{\mathrm{T}})$.If $P = \widetilde{\mathcal{L}}(f)$, then P is a simple closed curve.

(64) Let f be a non constant standard special circular sequence and given g_1, g_2. Suppose g_1 is a right part of f from i_1 to i_2 and g_2 is a right part of f from i_1 to i_2. Then $g_1 = g_2$.

(65) Let f be a non constant standard special circular sequence and given g_1, g_2. Suppose g_1 is a left part of f from i_1 to i_2 and g_2 is a left part of f from i_1 to i_2. Then $g_1 = g_2$.

(66) Let f be a non constant standard special circular sequence and given g_1, g_2. Suppose $i_1 \neq i_2$ and g_1 is a right part of f from i_1 to i_2 and g_2 is a left part of f from i_1 to i_2. Then $g_1(2) \neq g_2(2)$.

(67) Let f be a non constant standard special circular sequence and given g_1, g_2. Suppose $i_1 \neq i_2$ and g_1 is a part of f from i_1 to i_2 and g_2 is a part of f from i_1 to i_2 and $g_1(2) = g_2(2)$. Then $g_1 = g_2$.

Let f be a non constant standard special circular sequence and let i_1, i_2 be natural numbers. Let us assume that $1 \leqslant i_1$ and $i_1 + 1 \leqslant \operatorname{len} f$ and $1 \leqslant i_2$ and $i_2 + 1 \leqslant \operatorname{len} f$ and $i_1 \neq i_2$. The functor $\mathrm{Lower}(f, i_1, i_2)$ yields a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by the conditions (Def. 5).

(Def. 5)(i) $\mathrm{Lower}(f, i_1, i_2)$ is a part of f from i_1 to i_2,

(ii) if $(\pi_{i_1+1} f)_{\mathbf{1}} < (\pi_{i_1} f)_{\mathbf{1}}$ or $(\pi_{i_1+1} f)_{\mathbf{2}} < (\pi_{i_1} f)_{\mathbf{2}}$, then $(\mathrm{Lower}(f, i_1, i_2))(2) = f(i_1 + 1)$, and

(iii) if $(\pi_{i_1+1} f)_{\mathbf{1}} \geqslant (\pi_{i_1} f)_{\mathbf{1}}$ and $(\pi_{i_1+1} f)_{\mathbf{2}} \geqslant (\pi_{i_1} f)_{\mathbf{2}}$, then $(\mathrm{Lower}(f, i_1, i_2))(2) = f(\mathrm{S\_Drop}(i_1 -' 1, f))$.

The functor $\mathrm{Upper}(f, i_1, i_2)$ yielding a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the conditions (Def. 6).

(Def. 6)(i) $\mathrm{Upper}(f, i_1, i_2)$ is a part of f from i_1 to i_2,

(ii) if $(\pi_{i_1+1} f)_{\mathbf{1}} > (\pi_{i_1} f)_{\mathbf{1}}$ or $(\pi_{i_1+1} f)_{\mathbf{2}} > (\pi_{i_1} f)_{\mathbf{2}}$, then $(\mathrm{Upper}(f, i_1, i_2))(2) = f(i_1 + 1)$, and

(iii) if $(\pi_{i_1+1} f)_{\mathbf{1}} \leqslant (\pi_{i_1} f)_{\mathbf{1}}$ and $(\pi_{i_1+1} f)_{\mathbf{2}} \leqslant (\pi_{i_1} f)_{\mathbf{2}}$, then $(\mathrm{Upper}(f, i_1, i_2))(2) = f(\mathrm{S\_Drop}(i_1 -' 1, f))$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Czesław Byliński. Some properties of restrictions of finite sequences. *Formalized Mathematics*, 5(**2**):241–245, 1996.
[5] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[6] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.
[7] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Simple closed curves. *Formalized Mathematics*, 2(**5**):663–664, 1991.
[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[9] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Formalized Mathematics*, 3(**2**):275–278, 1992.
[10] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board - part I. *Formalized Mathematics*, 3(**1**):107–115, 1992.
[11] Yatsuka Nakamura and Jarosław Kotowicz. Connectedness conditions using polygonal arcs. *Formalized Mathematics*, 3(**1**):101–106, 1992.
[12] Yatsuka Nakamura and Roman Matuszewski. Reconstructions of special sequences. *Formalized Mathematics*, 6(**2**):255–263, 1997.
[13] Yatsuka Nakamura and Andrzej Trybulec. Decomposing a Go-board into cells. *Formalized Mathematics*, 5(**3**):323–328, 1996.
[14] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.
[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[17] Andrzej Trybulec. On the decomposition of finite sequences. *Formalized Mathematics*, 5(**3**):317–322, 1996.
[18] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[19] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[20] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received May 12, 1997