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Bounding Boxes for Compact Sets in $\mathcal{E}^2$

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Summary. We define pseudocompact topological spaces and prove that every compact space is pseudocompact. We also solve an exercise from [16] p.225 that the for a topological space X the following are equivalent:

- Every continuous real map from X is bounded (i.e. X is pseudocompact).
- Every continuous real map from X attains minimum.
- Every continuous real map from X attains maximum.

Finally, for a compact set in E^2 we define its bounding rectangle and introduce a collection of notions associated with the box.

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The papers [25], [30], [19], [7], [29], [24], [18], [17], [27], [21], [23], [10], [1], [26], [31], [3], [4], [14], [12], [13], [11], [22], [15], [20], [6], [5], [2], [8], [9], and [28] provide the notation and terminology for this paper.

1. Preliminaries

Let X be a set. Let us observe that X has non empty elements if and only if:

(Def. 1) $0 \notin X$.

We introduce X is without zero as a synonym of X has non empty elements. We introduce X has zero as an antonym of X has non empty elements.

Let us observe that $\mathbb{R}$ has zero and $\mathbb{N}$ has zero.

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Let us observe that there exists a set which is non empty and without zero and there exists a set which is non empty and has zero.

Let us observe that there exists a subset of $\mathbb{R}$ which is non empty and without zero and there exists a subset of $\mathbb{R}$ which is non empty and has zero.

Next we state the proposition

(1) For every set F such that F is non empty and $\subseteq$-linear and has non empty elements holds F is centered.

Let F be a set. Note that every family of subsets of F which is non empty and $\subseteq$-linear and has non empty elements is also centered.

Let A, B be sets and let f be a function from A into B. Then $\operatorname{rng} f$ is a subset of B.

Let X, Y be non empty sets and let f be a function from X into Y. Note that $f^\circ X$ is non empty.

Let X, Y be sets and let f be a function from X into Y. The functor ${}^{-1}f$ yields a function from 2^Y into 2^X and is defined by:

(Def. 2) For every subset y of Y holds $({}^{-1}f)(y) = f^{-1}(y)$.

We now state the proposition

(2) Let X, Y, x be sets, S be a subset of 2^Y, and f be a function from X into Y. If $x \in \bigcap(({}^{-1}f)^\circ S)$, then $f(x) \in \bigcap S$.

We follow the rules: p, q, r, r_1, r_2, s, t are real numbers, s_1 is a sequence of real numbers, and X, Y are subsets of $\mathbb{R}$.

One can prove the following propositions:

(3) If $|r| + |s| = 0$, then $r = 0$ and $s = 0$.

(4) If $r < s$ and $s < t$, then $|s| < |r| + |t|$.

(5) If $-s < r$ and $r < s$, then $|r| < s$.

(6) If s_1 is convergent and non-zero and $\lim s_1 = 0$, then ${s_1}^{-1}$ is non bounded.

(7) $\operatorname{rng} s_1$ is bounded iff s_1 is bounded.

Next we state four propositions:

(8) Let X be a non empty subset of $\mathbb{R}$ and given r. Suppose X is lower bounded. Then $r = \inf X$ if and only if the following conditions are satisfied:

(i) for every p such that $p \in X$ holds $p \geqslant r$, and

(ii) for every q such that for every p such that $p \in X$ holds $p \geqslant q$ holds $r \geqslant q$.

(9) Let X be a non empty subset of $\mathbb{R}$ and given r. Suppose X is upper bounded. Then $r = \sup X$ if and only if the following conditions are satisfied:

(i) for every p such that $p \in X$ holds $p \leqslant r$, and

(ii) for every q such that for every p such that $p \in X$ holds $p \leqslant q$ holds $r \leqslant q$.

(10) For every non empty subset X of $\mathbb{R}$ and for every subset Y of $\mathbb{R}$ such that $X \subseteq Y$ and Y is lower bounded holds $\inf Y \leqslant \inf X$.

(11) For every non empty subset X of $\mathbb{R}$ and for every subset Y of $\mathbb{R}$ such that $X \subseteq Y$ and Y is upper bounded holds $\sup X \leqslant \sup Y$.

Let X be a subset of $\mathbb{R}$. We say that X has maximum if and only if:

(Def. 3) X is upper bounded and $\sup X \in X$.

We say that X has minimum if and only if:

(Def. 4) X is lower bounded and $\inf X \in X$.

One can verify that there exists a subset of $\mathbb{R}$ which is non empty, closed, and bounded.

Let R be a family of subsets of $\mathbb{R}$. We say that R is open if and only if:

(Def. 5) For every subset X of $\mathbb{R}$ such that $X \in R$ holds X is open.

We say that R is closed if and only if:

(Def. 6) For every subset X of $\mathbb{R}$ such that $X \in R$ holds X is closed.

Let X be a subset of $\mathbb{R}$. The functor $-X$ yielding a subset of $\mathbb{R}$ is defined by:

(Def. 7) $-X = \{-r : r \in X\}$.

Next we state the proposition

(12) $r \in X$ iff $-r \in -X$.

Let X be a non empty subset of $\mathbb{R}$. One can check that $-X$ is non empty.

One can prove the following propositions:

(13) $--X = X$.

(14) X is upper bounded iff $-X$ is lower bounded.

(15) X is lower bounded iff $-X$ is upper bounded.

(16) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded holds $\inf X = -\sup(-X)$.

(17) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded holds $\sup X = -\inf(-X)$.

(18) X is closed iff $-X$ is closed.

Let X be a subset of $\mathbb{R}$ and let p be a real number. The functor $p+X$ yields a subset of $\mathbb{R}$ and is defined by:

(Def. 8) $p + X = \{p + r : r \in X\}$.

One can prove the following proposition

(19) $r \in X$ iff $s + r \in s + X$.

Let X be a non empty subset of $\mathbb{R}$ and let s be a real number. Observe that $s + X$ is non empty.

One can prove the following propositions:

(20) $X = 0 + X$.

(21) $s + (t + X) = (s + t) + X$.

(22) X is upper bounded iff $s + X$ is upper bounded.

(23) X is lower bounded iff $s + X$ is lower bounded.

(24) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded holds $\inf(s + X) = s + \inf X$.

(25) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded holds $\sup(s + X) = s + \sup X$.

(26) X is closed iff $s + X$ is closed.

Let X be a subset of $\mathbb{R}$. The functor $\operatorname{Inv} X$ yielding a subset of $\mathbb{R}$ is defined by:

(Def. 9) $\operatorname{Inv} X = \{\frac{1}{r} : r \in X\}$.

The following proposition is true

(27) For every without zero subset X of $\mathbb{R}$ such that $r \neq 0$ holds $r \in X$ iff $\frac{1}{r} \in \operatorname{Inv} X$.

Let X be a non empty without zero subset of $\mathbb{R}$. One can verify that $\operatorname{Inv} X$ is non empty and without zero.

Let X be a without zero subset of $\mathbb{R}$. One can verify that $\operatorname{Inv} X$ is without zero.

The following propositions are true:

(28) For every without zero subset X of $\mathbb{R}$ holds $\operatorname{Inv} \operatorname{Inv} X = X$.

(29) For every without zero subset X of $\mathbb{R}$ such that X is closed and bounded holds $\operatorname{Inv} X$ is closed.

(30) For every family Z of subsets of $\mathbb{R}$ such that Z is closed holds $\bigcap Z$ is closed.

Let X be a subset of $\mathbb{R}$. The functor $\overline{X}$ yielding a subset of $\mathbb{R}$ is defined by:

(Def. 10) $\overline{X} = \bigcap\{A, A$ ranges over elements of $2^{\mathbb{R}}$: $X \subseteq A \ \wedge \ A$ is closed$\}$.

Let X be a subset of $\mathbb{R}$. Observe that $\overline{X}$ is closed.

Next we state several propositions:

(31) For every closed subset Y of $\mathbb{R}$ such that $X \subseteq Y$ holds $\overline{X} \subseteq Y$.

(32) $X \subseteq \overline{X}$.

(33) X is closed iff $X = \overline{X}$.

(34) $\overline{\emptyset_{\mathbb{R}}} = \emptyset$.

(35) $\overline{\Omega_{\mathbb{R}}} = \mathbb{R}$.

(36) $\overline{X} = \overline{\overline{X}}$.

(37) If $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$.

(38) $r \in \overline{X}$ iff for every open subset O of $\mathbb{R}$ such that $r \in O$ holds $O \cap X$ is non empty.

(39) If $r \in \overline{X}$, then there exists s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 = r$.

2. Functions into Reals

Let A be a set, let f be a function from A into $\mathbb{R}$, and let a be a set. Then $f(a)$ is a real number.

Let X be a set and let f be a function from X into $\mathbb{R}$. We say that f is lower bounded if and only if:

(Def. 11) $f^\circ X$ is lower bounded.

We say that f is upper bounded if and only if:

(Def. 12) $f^\circ X$ is upper bounded.

Let X be a set and let f be a function from X into $\mathbb{R}$. We say that f is bounded if and only if:

(Def. 13) f is lower bounded and upper bounded.

We say that f has maximum if and only if:

(Def. 14) $f^\circ X$ has maximum.

We say that f has minimum if and only if:

(Def. 15) $f^\circ X$ has minimum.

Let X be a set. One can check that every function from X into $\mathbb{R}$ which is bounded is also lower bounded and upper bounded and every function from X into $\mathbb{R}$ which is lower bounded and upper bounded is also bounded.

Let X be a set and let f be a function from X into $\mathbb{R}$. The functor $-f$ yields a function from X into $\mathbb{R}$ and is defined as follows:

(Def. 16) For every set p such that $p \in X$ holds $(-f)(p) = -f(p)$.

The following propositions are true:

(40) For all sets X, A and for every function f from X into $\mathbb{R}$ holds $(-f)^\circ A = -f^\circ A$.

(41) For every set X and for every function f from X into $\mathbb{R}$ holds $--f = f$.

(42) For every non empty set X and for every function f from X into $\mathbb{R}$ holds f has minimum iff $-f$ has maximum.

(43) For every non empty set X and for every function f from X into $\mathbb{R}$ holds f has maximum iff $-f$ has minimum.

(44) For every set X and for every subset A of $\mathbb{R}$ and for every function f from X into $\mathbb{R}$ holds $(-f)^{-1}(A) = f^{-1}(-A)$.

Let X be a set, let r be a real number, and let f be a function from X into $\mathbb{R}$. The functor $r + f$ yielding a function from X into $\mathbb{R}$ is defined as follows:

(Def. 17) For every set p such that $p \in X$ holds $(r + f)(p) = r + f(p)$.

One can prove the following two propositions:

(45) For all sets X, A and for every function f from X into $\mathbb{R}$ and for every real number s holds $(s+f)^{\circ}A = s + f^{\circ}A$.

(46) For every set X and for every subset A of $\mathbb{R}$ and for every function f from X into $\mathbb{R}$ and for every s holds $(s+f)^{-1}(A) = f^{-1}(-s+A)$.

Let X be a set and let f be a function from X into $\mathbb{R}$. The functor $\mathrm{Inv}\, f$ yields a function from X into $\mathbb{R}$ and is defined by:

(Def. 18) For every set p such that $p \in X$ holds $(\mathrm{Inv}\, f)(p) = \frac{1}{f(p)}$.

We now state the proposition

(47) Let X be a set, A be a without zero subset of $\mathbb{R}$, and f be a function from X into $\mathbb{R}$. If $0 \notin \mathrm{rng}\, f$, then $(\mathrm{Inv}\, f)^{-1}(A) = f^{-1}(\mathrm{Inv}\, A)$.

3. Real maps

Let T be a 1-sorted structure.

(Def. 19) A function from the carrier of T into $\mathbb{R}$ is called a real map of T.

Let T be a non empty 1-sorted structure. Note that there exists a real map of T which is bounded.

In this article we present several logical schemes. The scheme *NonUniqExRF* deals with a non empty topological structure $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a real map f of $\mathcal{A}$ such that for every element x of the carrier of $\mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the parameters meet the following requirement:

- For every set x such that $x \in$ the carrier of $\mathcal{A}$ there exists r such that $\mathcal{P}[x, r]$.

The scheme *LambdaRF* deals with a non empty topological structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a real number, and states that:

There exists a real map f of $\mathcal{A}$ such that for every element x of the carrier of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

Let T be a 1-sorted structure, let f be a real map of T, and let P be a set. Then $f^{-1}(P)$ is a subset of T.

Let T be a 1-sorted structure and let f be a real map of T. The functor $\inf f$ yielding a real number is defined by:

(Def. 20) $\inf f = \inf(f^{\circ}(\text{the carrier of } T))$.

The functor $\sup f$ yields a real number and is defined by:

(Def. 21) $\sup f = \sup(f^{\circ}(\text{the carrier of } T))$.

Next we state three propositions:

(48) Let T be a non empty topological space and f be a lower bounded real map of T. Then $r = \inf f$ if and only if the following conditions are satisfied:

(i) for every point p of T holds $f(p) \geqslant r$, and

(ii) for every real number q such that for every point p of T holds $f(p) \geqslant q$ holds $r \geqslant q$.

(49) Let T be a non empty topological space and f be an upper bounded real map of T. Then $r = \sup f$ if and only if the following conditions are satisfied:

(i) for every point p of T holds $f(p) \leqslant r$, and

(ii) for every real number q such that for every point p of T holds $f(p) \leqslant q$ holds $r \leqslant q$.

(50) For every non empty 1-sorted structure T and for every bounded real map f of T holds $\inf f \leqslant \sup f$.

Let T be a 1-sorted structure and let f be a real map of T. The functor $-f$ yielding a real map of T is defined by:

(Def. 22) $-f = -f$.

Let T be a 1-sorted structure, let r be a real number, and let f be a real map of T. The functor $r + f$ yields a real map of T and is defined by:

(Def. 23) $r + f = r + f$.

Let T be a 1-sorted structure and let f be a real map of T. The functor $\operatorname{Inv} f$ yields a real map of T and is defined by:

(Def. 24) $\operatorname{Inv} f = \operatorname{Inv} f$.

Let T be a topological structure and let f be a real map of T. We say that f is continuous if and only if:

(Def. 25) For every subset Y of $\mathbb{R}$ such that Y is closed holds $f^{-1}(Y)$ is closed.

Let T be a non empty topological space. Note that there exists a real map of T which is continuous.

Let T be a non empty topological space and let S be a non empty subspace of T. One can check that there exists a real map of S which is continuous.

In the sequel T is a topological space and f is a real map of T.

Next we state several propositions:

(51) f is continuous iff for every subset Y of $\mathbb{R}$ such that Y is open holds $f^{-1}(Y)$ is open.

(52) If f is continuous, then $-f$ is continuous.

(53) If f is continuous, then $r + f$ is continuous.

(54) If f is continuous and $0 \notin \operatorname{rng} f$, then $\operatorname{Inv} f$ is continuous.

(55) For every family R of subsets of $\mathbb{R}$ such that f is continuous and R is open holds $({}^{-1}f)^{\circ}R$ is open.

(56) For every family R of subsets of $\mathbb{R}$ such that f is continuous and R is closed holds $({}^{-1}f)^{\circ}R$ is closed.

Let T be a non empty topological space, let X be a subset of T, and let f be a real map of T. The functor $f \restriction X$ yielding a real map of $T{\restriction}X$ is defined as follows:

(Def. 26) $f \restriction X = f{\restriction}X$.

Let T be a non empty topological space. One can check that there exists a subset of T which is compact and non empty.

Let T be a non empty topological space, let f be a continuous real map of T, and let X be a compact non empty subset of T. Note that $f \restriction X$ is continuous.

Let T be a non empty topological space and let P be a compact non empty subset of T. Note that $T{\restriction}P$ is compact.

4. Pseudocompact spaces

We now state two propositions:

(57) Let T be a non empty topological space. Then for every real map f of T such that f is continuous holds f has maximum if and only if for every real map f of T such that f is continuous holds f has minimum.

(58) Let T be a non empty topological space. Then for every real map f of T such that f is continuous holds f is bounded if and only if for every real map f of T such that f is continuous holds f has maximum.

Let T be a topological space. We say that T is pseudocompact if and only if:

(Def. 27) For every real map f of T such that f is continuous holds f is bounded.

Let us mention that every non empty topological space which is compact is also pseudocompact.

Let us mention that there exists a topological space which is compact and non empty.

Let T be a pseudocompact non empty topological space. One can check that every real map of T which is continuous is also bounded and has maximum and minimum.

We now state two propositions:

(59) Let T be a non empty topological space, X, Y be non empty compact subsets of T, and f be a continuous real map of T. If $X \subseteq Y$, then $\inf(f \restriction Y) \leqslant \inf(f \restriction X)$.

(60) Let T be a non empty topological space, X, Y be non empty compact subsets of T, and f be a continuous real map of T. If $X \subseteq Y$, then $\sup(f \restriction X) \leqslant \sup(f \restriction Y)$.

5. Bounding boxes for compact sets in $\mathcal{E}^2$

Let n be a natural number and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Note that $\mathcal{L}(p_1, p_2)$ is compact.

One can prove the following proposition

(61) For every natural number n and for all compact subsets X, Y of $\mathcal{E}^n_{\mathrm{T}}$ holds $X \cap Y$ is compact.

In the sequel p is a point of $\mathcal{E}^2_{\mathrm{T}}$, P is a subset of $\mathcal{E}^2_{\mathrm{T}}$, and X is a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$.

The real map proj1 of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 28) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\mathrm{proj1})(p) = p_{\mathbf{1}}$.

The real map proj2 of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 29) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\mathrm{proj2})(p) = p_{\mathbf{2}}$.

One can prove the following propositions:

(62) $(\mathrm{proj1})^{-1}(]r, s[) = \{[r_1, r_2] : r < r_1 \ \wedge \ r_1 < s\}$.

(63) For all r, s such that $P = \{[r_1, r_2] : r < r_1 \ \wedge \ r_1 < s\}$ holds P is open.

(64) $(\mathrm{proj2})^{-1}(]r, s[) = \{[r_1, r_2] : r < r_2 \ \wedge \ r_2 < s\}$.

(65) For all r, s such that $P = \{[r_1, r_2] : r < r_2 \ \wedge \ r_2 < s\}$ holds P is open.

One can verify that proj1 is continuous and proj2 is continuous.

One can prove the following two propositions:

(66) For every non empty subset X of $\mathcal{E}^2_{\mathrm{T}}$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ holds $(\mathrm{proj1} \restriction X)(p) = p_{\mathbf{1}}$.

(67) For every non empty subset X of $\mathcal{E}^2_{\mathrm{T}}$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ holds $(\mathrm{proj2} \restriction X)(p) = p_{\mathbf{2}}$.

Let X be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor W-bound X yielding a real number is defined by:

(Def. 30) W-bound $X = \inf(\mathrm{proj1} \restriction X)$.

The functor N-bound X yielding a real number is defined as follows:

(Def. 31) N-bound $X = \sup(\mathrm{proj2} \restriction X)$.

The functor E-bound X yielding a real number is defined by:

(Def. 32) E-bound $X = \sup(\mathrm{proj1} \restriction X)$.

The functor S-bound X yielding a real number is defined by:

(Def. 33) S-bound $X = \inf(\mathrm{proj2} \restriction X)$.

We now state the proposition

(68) If $p \in X$, then W-bound $X \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant$ E-bound X and S-bound $X \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant$ N-bound X.

Let X be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor SW-corner X yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 34) SW-corner $X = [$W-bound X, S-bound $X]$.

The functor NW-corner X yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 35) NW-corner $X = [$W-bound X, N-bound $X]$.

The functor NE-corner X yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 36) NE-corner $X = [$E-bound X, N-bound $X]$.

The functor SE-corner X yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 37) SE-corner $X = [$E-bound X, S-bound $X]$.

Let X be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor W-most X yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 38) W-most $X = \mathcal{L}($SW-corner X, NW-corner $X) \cap X$.

The functor N-most X yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 39) N-most $X = \mathcal{L}($NW-corner X, NE-corner $X) \cap X$.

The functor E-most X yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 40) E-most $X = \mathcal{L}($SE-corner X, NE-corner $X) \cap X$.

The functor S-most X yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 41) S-most $X = \mathcal{L}($SW-corner X, SE-corner $X) \cap X$.

Let X be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. One can check the following observations:

* W-most X is non empty and compact,
* N-most X is non empty and compact,
* E-most X is non empty and compact, and
* S-most X is non empty and compact.

Let X be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor W-min X yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 42) W-min $X = [$W-bound $X, \inf($proj2 $\restriction$ W-most $X)]$.

The functor W-max X yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 43) W-max $X = [$W-bound $X, \sup($proj2 $\restriction$ W-most $X)]$.

The functor N-min X yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 44) N-min $X = [\inf($proj1 $\restriction$ N-most $X)$, N-bound $X]$.

The functor N-max X yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 45) N-max $X = [\sup($proj1 $\restriction$ N-most $X)$, N-bound $X]$.

The functor E-max X yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 46) $\text{E-max}\,X = [\text{E-bound}\,X, \sup(\text{proj2} \restriction \text{E-most}\,X)]$.

The functor $\text{E-min}\,X$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 47) $\text{E-min}\,X = [\text{E-bound}\,X, \inf(\text{proj2} \restriction \text{E-most}\,X)]$.

The functor $\text{S-max}\,X$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 48) $\text{S-max}\,X = [\sup(\text{proj1} \restriction \text{S-most}\,X), \text{S-bound}\,X]$.

The functor $\text{S-min}\,X$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 49) $\text{S-min}\,X = [\inf(\text{proj1} \restriction \text{S-most}\,X), \text{S-bound}\,X]$.

Next we state a number of propositions:

(69) $(\text{SW-corner}\,X)_{\mathbf{1}} = \text{W-bound}\,X$ and $(\text{W-min}\,X)_{\mathbf{1}} = \text{W-bound}\,X$ and $(\text{W-max}\,X)_{\mathbf{1}} = \text{W-bound}\,X$ and $(\text{NW-corner}\,X)_{\mathbf{1}} = \text{W-bound}\,X$.

(70) $(\text{SW-corner}\,X)_{\mathbf{1}} = (\text{NW-corner}\,X)_{\mathbf{1}}$ and $(\text{SW-corner}\,X)_{\mathbf{1}} = (\text{W-min}\,X)_{\mathbf{1}}$ and $(\text{SW-corner}\,X)_{\mathbf{1}} = (\text{W-max}\,X)_{\mathbf{1}}$ and $(\text{W-min}\,X)_{\mathbf{1}} = (\text{W-max}\,X)_{\mathbf{1}}$ and $(\text{W-min}\,X)_{\mathbf{1}} = (\text{NW-corner}\,X)_{\mathbf{1}}$ and $(\text{W-max}\,X)_{\mathbf{1}} = (\text{NW-corner}\,X)_{\mathbf{1}}$.

(71) $(\text{SW-corner}\,X)_{\mathbf{2}} = \text{S-bound}\,X$ and $(\text{W-min}\,X)_{\mathbf{2}} = \inf(\text{proj2} \restriction \text{W-most}\,X)$ and $(\text{W-max}\,X)_{\mathbf{2}} = \sup(\text{proj2} \restriction \text{W-most}\,X)$ and $(\text{NW-corner}\,X)_{\mathbf{2}} = \text{N-bound}\,X$.

(72) $(\text{SW-corner}\,X)_{\mathbf{2}} \leqslant (\text{W-min}\,X)_{\mathbf{2}}$ and $(\text{SW-corner}\,X)_{\mathbf{2}} \leqslant (\text{W-max}\,X)_{\mathbf{2}}$ and $(\text{SW-corner}\,X)_{\mathbf{2}} \leqslant (\text{NW-corner}\,X)_{\mathbf{2}}$ and $(\text{W-min}\,X)_{\mathbf{2}} \leqslant (\text{W-max}\,X)_{\mathbf{2}}$ and $(\text{W-min}\,X)_{\mathbf{2}} \leqslant (\text{NW-corner}\,X)_{\mathbf{2}}$ and $(\text{W-max}\,X)_{\mathbf{2}} \leqslant (\text{NW-corner}\,X)_{\mathbf{2}}$.

(73) If $p \in \text{W-most}\,X$, then $p_{\mathbf{1}} = (\text{W-min}\,X)_{\mathbf{1}}$ and $(\text{W-min}\,X)_{\mathbf{2}} \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant (\text{W-max}\,X)_{\mathbf{2}}$.

(74) $\text{W-most}\,X \subseteq \mathcal{L}(\text{W-min}\,X, \text{W-max}\,X)$.

(75) $\mathcal{L}(\text{W-min}\,X, \text{W-max}\,X) \subseteq \mathcal{L}(\text{SW-corner}\,X, \text{NW-corner}\,X)$.

(76) $\text{W-min}\,X \in \text{W-most}\,X$ and $\text{W-max}\,X \in \text{W-most}\,X$.

(77) $\mathcal{L}(\text{SW-corner}\,X, \text{W-min}\,X) \cap X = \{\text{W-min}\,X\}$ and $\mathcal{L}(\text{W-max}\,X, \text{NW-corner}\,X) \cap X = \{\text{W-max}\,X\}$.

(78) If $\text{W-min}\,X = \text{W-max}\,X$, then $\text{W-most}\,X = \{\text{W-min}\,X\}$.

(79) $(\text{NW-corner}\,X)_{\mathbf{2}} = \text{N-bound}\,X$ and $(\text{N-min}\,X)_{\mathbf{2}} = \text{N-bound}\,X$ and $(\text{N-max}\,X)_{\mathbf{2}} = \text{N-bound}\,X$ and $(\text{NE-corner}\,X)_{\mathbf{2}} = \text{N-bound}\,X$.

(80) $(\text{NW-corner}\,X)_{\mathbf{2}} = (\text{NE-corner}\,X)_{\mathbf{2}}$ and $(\text{NW-corner}\,X)_{\mathbf{2}} = (\text{N-min}\,X)_{\mathbf{2}}$ and $(\text{NW-corner}\,X)_{\mathbf{2}} = (\text{N-max}\,X)_{\mathbf{2}}$ and $(\text{N-min}\,X)_{\mathbf{2}} = (\text{N-max}\,X)_{\mathbf{2}}$ and $(\text{N-min}\,X)_{\mathbf{2}} = (\text{NE-corner}\,X)_{\mathbf{2}}$ and $(\text{N-max}\,X)_{\mathbf{2}} = (\text{NE-corner}\,X)_{\mathbf{2}}$.

(81) $(\text{NW-corner}\,X)_{\mathbf{1}} = \text{W-bound}\,X$ and $(\text{N-min}\,X)_{\mathbf{1}} = \inf(\text{proj1} \restriction \text{N-most}\,X)$ and $(\text{N-max}\,X)_{\mathbf{1}} = \sup(\text{proj1} \restriction \text{N-most}\,X)$ and $(\text{NE-corner}\,X)_{\mathbf{1}} = \text{E-bound}\,X$.

(82) $(\text{NW-corner}\,X)_{\mathbf{1}} \leqslant (\text{N-min}\,X)_{\mathbf{1}}$ and $(\text{NW-corner}\,X)_{\mathbf{1}} \leqslant (\text{N-max}\,X)_{\mathbf{1}}$ and $(\text{NW-corner}\,X)_{\mathbf{1}} \leqslant (\text{NE-corner}\,X)_{\mathbf{1}}$ and $(\text{N-min}\,X)_{\mathbf{1}} \leqslant (\text{N-max}\,X)_{\mathbf{1}}$ and $(\text{N-min}\,X)_{\mathbf{1}} \leqslant (\text{NE-corner}\,X)_{\mathbf{1}}$ and $(\text{N-max}\,X)_{\mathbf{1}} \leqslant (\text{NE-corner}\,X)_{\mathbf{1}}$.

(83) If $p \in$ N-most X, then $p_{\mathbf{2}} =$ (N-min $X)_{\mathbf{2}}$ and (N-min $X)_{\mathbf{1}} \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant$ (N-max $X)_{\mathbf{1}}$.

(84) N-most $X \subseteq \mathcal{L}$(N-min X, N-max X).

(85) $\mathcal{L}$(N-min X, N-max X) $\subseteq \mathcal{L}$(NW-corner X, NE-corner X).

(86) N-min $X \in$ N-most X and N-max $X \in$ N-most X.

(87) $\mathcal{L}$(NW-corner X, N-min X) $\cap X = \{$N-min $X\}$ and $\mathcal{L}$(N-max X, NE-corner X) $\cap X = \{$N-max $X\}$.

(88) If N-min $X =$ N-max X, then N-most $X = \{$N-min $X\}$.

(89) (SE-corner $X)_{\mathbf{1}} =$ E-bound X and (E-min $X)_{\mathbf{1}} =$ E-bound X and (E-max $X)_{\mathbf{1}} =$ E-bound X and (NE-corner $X)_{\mathbf{1}} =$ E-bound X.

(90) (SE-corner $X)_{\mathbf{1}} =$ (NE-corner $X)_{\mathbf{1}}$ and (SE-corner $X)_{\mathbf{1}} =$ (E-min $X)_{\mathbf{1}}$ and (SE-corner $X)_{\mathbf{1}} =$ (E-max $X)_{\mathbf{1}}$ and (E-min $X)_{\mathbf{1}} =$ (E-max $X)_{\mathbf{1}}$ and (E-min $X)_{\mathbf{1}} =$ (NE-corner $X)_{\mathbf{1}}$ and (E-max $X)_{\mathbf{1}} =$ (NE-corner $X)_{\mathbf{1}}$.

(91) (SE-corner $X)_{\mathbf{2}} =$ S-bound X and (E-min $X)_{\mathbf{2}} =$ inf(proj2 $\restriction$ E-most X) and (E-max $X)_{\mathbf{2}} =$ sup(proj2 $\restriction$ E-most X) and (NE-corner $X)_{\mathbf{2}} =$ N-bound X.

(92) (SE-corner $X)_{\mathbf{2}} \leqslant$ (E-min $X)_{\mathbf{2}}$ and (SE-corner $X)_{\mathbf{2}} \leqslant$ (E-max $X)_{\mathbf{2}}$ and (SE-corner $X)_{\mathbf{2}} \leqslant$ (NE-corner $X)_{\mathbf{2}}$ and (E-min $X)_{\mathbf{2}} \leqslant$ (E-max $X)_{\mathbf{2}}$ and (E-min $X)_{\mathbf{2}} \leqslant$ (NE-corner $X)_{\mathbf{2}}$ and (E-max $X)_{\mathbf{2}} \leqslant$ (NE-corner $X)_{\mathbf{2}}$.

(93) If $p \in$ E-most X, then $p_{\mathbf{1}} =$ (E-min $X)_{\mathbf{1}}$ and (E-min $X)_{\mathbf{2}} \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant$ (E-max $X)_{\mathbf{2}}$.

(94) E-most $X \subseteq \mathcal{L}$(E-min X, E-max X).

(95) $\mathcal{L}$(E-min X, E-max X) $\subseteq \mathcal{L}$(SE-corner X, NE-corner X).

(96) E-min $X \in$ E-most X and E-max $X \in$ E-most X.

(97) $\mathcal{L}$(SE-corner X, E-min X) $\cap X = \{$E-min $X\}$ and $\mathcal{L}$(E-max X, NE-corner X) $\cap X = \{$E-max $X\}$.

(98) If E-min $X =$ E-max X, then E-most $X = \{$E-min $X\}$.

(99) (SW-corner $X)_{\mathbf{2}} =$ S-bound X and (S-min $X)_{\mathbf{2}} =$ S-bound X and (S-max $X)_{\mathbf{2}} =$ S-bound X and (SE-corner $X)_{\mathbf{2}} =$ S-bound X.

(100) (SW-corner $X)_{\mathbf{2}} =$ (SE-corner $X)_{\mathbf{2}}$ and (SW-corner $X)_{\mathbf{2}} =$ (S-min $X)_{\mathbf{2}}$ and (SW-corner $X)_{\mathbf{2}} =$ (S-max $X)_{\mathbf{2}}$ and (S-min $X)_{\mathbf{2}} =$ (S-max $X)_{\mathbf{2}}$ and (S-min $X)_{\mathbf{2}} =$ (SE-corner $X)_{\mathbf{2}}$ and (S-max $X)_{\mathbf{2}} =$ (SE-corner $X)_{\mathbf{2}}$.

(101) (SW-corner $X)_{\mathbf{1}} =$ W-bound X and (S-min $X)_{\mathbf{1}} =$ inf(proj1 $\restriction$ S-most X) and (S-max $X)_{\mathbf{1}} =$ sup(proj1 $\restriction$ S-most X) and (SE-corner $X)_{\mathbf{1}} =$ E-bound X.

(102) (SW-corner $X)_{\mathbf{1}} \leqslant$ (S-min $X)_{\mathbf{1}}$ and (SW-corner $X)_{\mathbf{1}} \leqslant$ (S-max $X)_{\mathbf{1}}$ and (SW-corner $X)_{\mathbf{1}} \leqslant$ (SE-corner $X)_{\mathbf{1}}$ and (S-min $X)_{\mathbf{1}} \leqslant$ (S-max $X)_{\mathbf{1}}$ and (S-min $X)_{\mathbf{1}} \leqslant$ (SE-corner $X)_{\mathbf{1}}$ and (S-max $X)_{\mathbf{1}} \leqslant$ (SE-corner $X)_{\mathbf{1}}$.

(103) If $p \in$ S-most X, then $p_{\mathbf{2}} = (\text{S-min}\, X)_{\mathbf{2}}$ and $(\text{S-min}\, X)_{\mathbf{1}} \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant (\text{S-max}\, X)_{\mathbf{1}}$.

(104) S-most $X \subseteq \mathcal{L}(\text{S-min}\, X, \text{S-max}\, X)$.

(105) $\mathcal{L}(\text{S-min}\, X, \text{S-max}\, X) \subseteq \mathcal{L}(\text{SW-corner}\, X, \text{SE-corner}\, X)$.

(106) S-min $X \in$ S-most X and S-max $X \in$ S-most X.

(107) $\mathcal{L}(\text{SW-corner}\, X, \text{S-min}\, X) \cap X = \{\text{S-min}\, X\}$ and $\mathcal{L}(\text{S-max}\, X, \text{SE-corner}\, X) \cap X = \{\text{S-max}\, X\}$.

(108) If S-min $X =$ S-max X, then S-most $X = \{\text{S-min}\, X\}$.

(109) If W-max $X =$ N-min X, then W-max $X =$ NW-corner X.

(110) If N-max $X =$ E-max X, then N-max $X =$ NE-corner X.

(111) If E-min $X =$ S-max X, then E-min $X =$ SE-corner X.

(112) If S-min $X =$ W-min X, then S-min $X =$ SW-corner X.

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